Application of Quantum Error Correction and Fault-Tolerance in Quantum Metrology

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2026 AlgoLab Summer School - Towards Fault Tolerant Quantum Computing,

AlgoLab - Institut quantique,

2:30PM-6PM , June 17, 2026

PERIMETER INSTITUTE

Outline

- Part 1: Introduction to Quantum Metrology
- Part 2: Heisenberg Limit vs Standard Quantum Limit

  (— — Coffee Break? — —)
- Part 3: Introduction to Quantum Error Correction
- Part 4: Quantum Error Correction for Quantum Metrology
  - Case Study I: Phase estimation under bit-flip noise
  - Case Study II: ZZZ interaction estimation under single-qubit noise
- Part 5: Fault Tolerance in Quantum Metrology (An Example)

Part 1

INTRODUCTION TO QUANTUM METROLOGY

* CLASSICAL PARAMETER ESTIMATION
* CRAMÉR-RAO BOUND
* QUANTUM PARAMETER ESTIMATION

QUANTUM METROLOGY

Quantum state → Quantum Measurement → Probability Distribution

$$\rho_\theta \longrightarrow \boxed{\{M_x\}_x} \longrightarrow p(x|\theta)$$

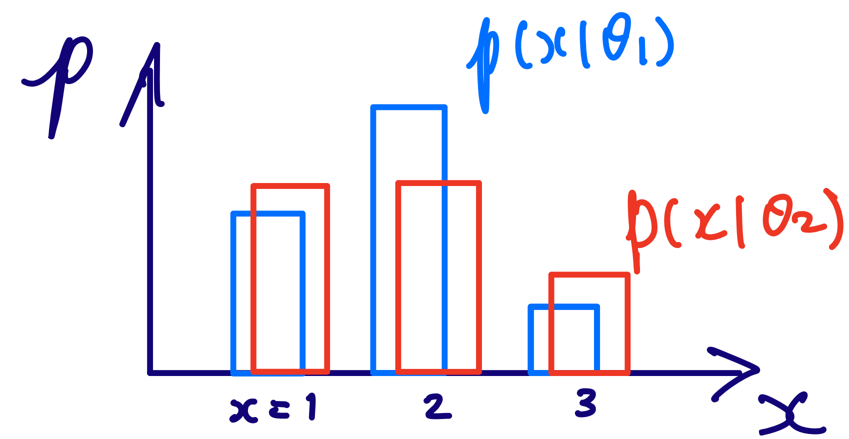
Density matrix

$\rho_\theta \geq 0, \quad \mathrm{Tr}(\rho_\theta) = 1,$

POVM

$M_x \geq 0, \quad \sum_x M_x = \mathbb{I}$

$$p(x|\theta) = \mathrm{Tr}(\rho_\theta M_x)$$



θ_1 v.s. θ_2

MEAN SQUARE ERROR

* Estimator $\hat{\theta}(x) : \mathcal{X} \longrightarrow \Theta$

(Domain of x) (Domain of θ)

$\hat{\theta}(x_1, \dots x_N) : \mathcal{X}^N \longrightarrow \Theta$

* Unbiased Estimator

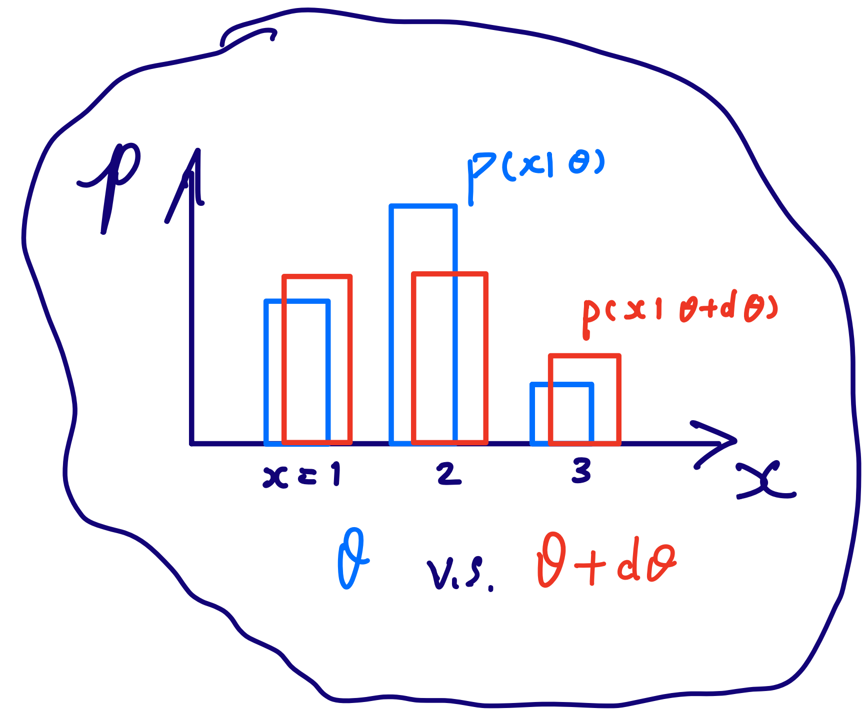
$$\mathbb{E}_x(\hat{\theta}) = \theta$$ — True Value of θ

* Mean Square Error (MSE)

$$V_\theta = \mathbb{E}_x[(\hat{\theta} - \theta)^2]$$ — Characterizes estimation precision

CRAMÉR-RAO BOUND

* Lower bound on MSE $V_\theta \geqslant \dfrac{1}{\sqrt{N \cdot F}}$



N: number of samples

F: (classical) Fisher information FI

$$F = \sum_x \frac{1}{p(x|\theta)} \left[\frac{\partial p(x|\theta)}{\partial \theta} \right]^2$$

* FI can be interpreted as the second-order derivative of FIDELITY

Fidelity: $f(p_\theta, p_{\theta+d\theta}) = \sum_x \sqrt{p(x|\theta)\, p(x|\theta+d\theta)}$

Taylor Expansion:

$$= 1 - \frac{(d\theta)^2}{8} \sum_x \frac{(\partial_\theta\, p(x|\theta))^2}{p(x|\theta)} + O(d\theta^3) = 1 - \frac{(d\theta)^2}{8} \cdot F + O(d\theta^3)$$

PROOF OF CRAMÉR-RAO BOUND

* Unbiased Estimator

$$\mathbb{E}_x(\hat{\theta}) = \theta \quad \text{— True Value of } \theta$$

* $$\begin{cases} \sum_x p(x|\theta)\,\hat{\theta}(x) = \theta \\ \sum_x p(x|\theta) = 1 \end{cases} \xRightarrow{\text{Derivative}} \begin{matrix} \sum_x \partial_\theta p(x|\theta)\,\hat{\theta}(x) = 1 \\ \sum_x \partial_\theta p(x|\theta) = 0 \end{matrix}$$

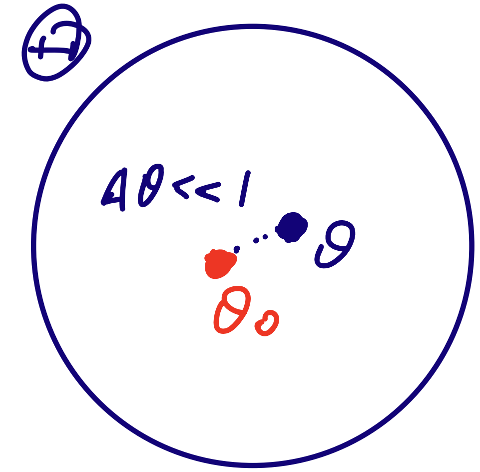
$$\Longrightarrow \sum_x \boxed{\frac{\partial_\theta p(x|\theta)}{\sqrt{p(x|\theta)}}} \cdot \boxed{\sqrt{p(x|\theta)}\,(\hat{\theta}(x) - \theta)} = 1$$

$$\xRightarrow{\text{Cauchy-Schwarz}} \sum_x \frac{(\partial_\theta p(x|\theta))^2}{p(x|\theta)} \times \sum_x p(x|\theta)(\hat{\theta}(x) - \theta)^2 \geq 1$$

$$FI \times V \geq 1$$

ATTAINABILITY OF CR BOUND

* Locally unbiased estimator at $\theta = \theta_0$:



$$\hat{\theta}_{l.u.}(x;\theta_0) = \theta_0 + \frac{1}{F(\theta_0)} \cdot \frac{\partial_\theta \, p(x|\theta)\big|_{\theta=\theta_0}}{p(x|\theta_0)}$$

Satisfying $\mathbb{E}_x(\hat{\theta}_{l.u.})\Big|_{\theta=\theta_0} = \theta_0$, and $V(\hat{\theta}_{l.u.})\Big|_{\theta=\theta_0} = \dfrac{1}{F(\theta_0)}$

$$\frac{\partial}{\partial\theta}\mathbb{E}_x(\hat{\theta}_{l.u.})\Big|_{\theta=\theta_0} = 1 .$$

(CR bound)

(locally unbiasedness condition)

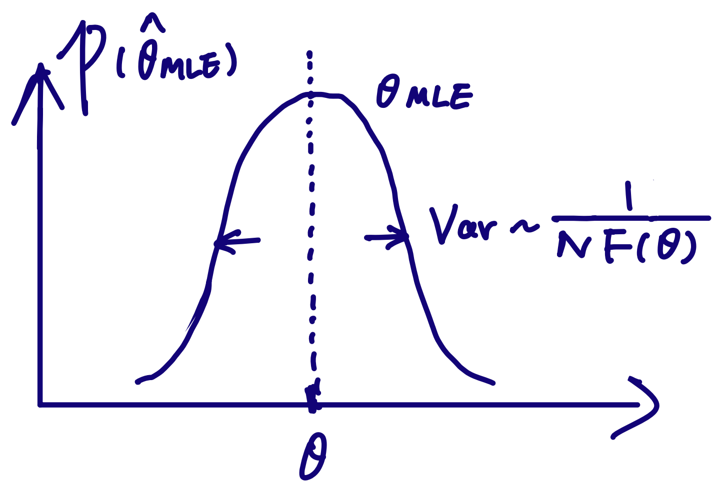
ATTAINABILITY OF CR BOUND

* Maximally likelihood estimation

$$\hat{\theta}_{MLE} = \underset{\theta}{\arg\max} \; p(x_1, x_2, \dots x_N | \theta)$$

* Under certain regularity conditions,

$$\sqrt{N}\,(\hat{\theta}_{MLE} - \theta) \xrightarrow{d} \text{Normal}(0, F(\theta)^{-1})$$



QUANTUM FISHER INFORMATION

* Classical FI

$$F(\{p(x|\theta)\}) = \sum_x \frac{1}{p(x|\theta)} \left[\frac{\partial p(x|\theta)}{\partial \theta} \right]^2 , \quad p(x|\theta) = \mathrm{Tr}(\rho_\theta M_x)$$

$$= \underbrace{\mathbb{E}_x}_{\text{Expectation}} \left[\left(\partial_\theta \log p(x|\theta) \right)^2 \right]$$

logarithmic derivative

* Quantum FI (QFI)

$$F(\rho_\theta) = \max_{\{M_x\}_x} F(p(x|\theta)) = \underbrace{\mathrm{Tr}(\rho_\theta L_\theta^2)}_{\text{Expectation}}$$

symmetric logarithmic derivative SLD

QUANTUM FISHER INFORMATION

* $\boxed{\partial_\theta \log p(x|\theta)} = \dfrac{\partial_\theta p(x|\theta)}{p(x|\theta)}$

$$\Downarrow$$

$$\frac{1}{2}\left(\underbrace{L_\theta}_{\text{SLD}} \rho_\theta + \rho_\theta L_\theta\right) = \partial_\theta \rho_\theta \quad , \; L_\theta \text{ Hermitian}$$

* QFI can be interpreted as the second-order derivative of FIDELITY

Fidelity: $f(\rho_\theta, \rho_{\theta+d\theta}) = \mathrm{Tr}\left(\sqrt{\sqrt{\rho_\theta}\, \rho_{\theta+d\theta} \sqrt{\rho_\theta}}\right)$

Taylor Expansion: $= 1 - \dfrac{(d\theta)^2}{8} \underbrace{F(\rho_\theta)}_{\text{QFI}} + O(d\theta^3)$

PROPERTIES OF QFI (FI)

* $\underset{\text{FI}}{F}(\{p(x|\theta)\}) = \underset{\text{QFI}}{F}(\rho_\theta)$, $\rho_\theta = \sum_x p(x|\theta)|x\rangle\langle x|$

* Additivity: $F(\rho_\theta \otimes \sigma_\theta) = F(\rho_\theta) + F(\sigma_\theta)$

$$\Longrightarrow \quad F(\rho_\theta^{\otimes n}) = n\, F(\rho_\theta)$$

* Monotonicity: $F(\underbrace{\mathcal{E}}_{\text{any parameter-independent quantum channel}}(\rho_\theta)) \leq F(\rho_\theta)$

* Convexity: $F(p_1 \rho_\theta + p_2 \sigma_\theta)$

$$\leq p_1 F(\rho_\theta) + p_2 F(\sigma_\theta)$$

QUANTUM FISHER INFORMATION

* Pure state QFI Calculation

$$\rho_\theta = |\psi_\theta\rangle\langle\psi_\theta| \ , \quad L_\theta = 2\left[|\partial_\theta\psi_\theta\rangle\langle\psi_\theta| + |\psi_\theta\rangle\langle\partial_\theta\psi_\theta|\right]$$

Verify SLD :

$$\frac{1}{2}\left[L_\theta\rho_\theta + \rho_\theta L_\theta\right] = |\partial_\theta\psi_\theta\rangle\langle\psi_\theta| + |\psi_\theta\rangle\langle\partial_\theta\psi_\theta| + |\psi_\theta\rangle\langle\psi_\theta| \left(\underbrace{\langle\psi_\theta|\partial_\theta\psi_\theta\rangle + \langle\partial_\theta\psi_\theta|\psi_\theta\rangle}_{= \partial_\theta(\langle\psi_\theta|\psi_\theta\rangle) = \partial_\theta 1 = 0} \right)$$

$$= |\partial_\theta\psi\rangle\langle\psi| + |\psi\rangle\langle\partial_\theta\psi| = \partial_\theta\left(|\psi\rangle\langle\psi|\right)$$

$$F(\psi_\theta) = \mathrm{Tr}(\rho_\theta L_\theta^2) = 4\langle\psi|\left[|\partial_\theta\psi\rangle\langle\psi|\partial_\theta\psi\rangle\langle\psi| + |\partial_\theta\psi\rangle\langle\psi|\psi\rangle\langle\partial_\theta\psi| + |\psi\rangle\langle\partial_\theta\psi|\psi\rangle\langle\partial_\theta\psi| + |\psi\rangle\langle\partial_\theta\psi|\partial_\theta\psi\rangle\langle\psi|\right]|\psi\rangle$$

$$= 4\left[\langle\psi|\partial_\theta\psi\rangle\langle\psi|\partial_\theta\psi\rangle + \langle\psi|\partial_\theta\psi\rangle\langle\partial_\theta\psi|\psi\rangle + \langle\partial_\theta\psi|\psi\rangle\langle\partial_\theta\psi|\psi\rangle + \langle\partial_\theta\psi|\partial_\theta\psi\rangle\right]$$

$$= 4\left[\langle\partial_\theta\psi|\partial_\theta\psi\rangle - \langle\psi|\partial_\theta\psi\rangle\langle\partial_\theta\psi|\psi\rangle\right]$$

QUANTUM FISHER INFORMATION

* Mixed state QFI Calculation

$$\rho_\theta = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i| \qquad \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_d \end{bmatrix}$$

functions of θ

$$\langle\psi_k|L_i|\psi_l\rangle = \begin{cases} \dfrac{2\langle\psi_k|\partial_\theta\rho_\theta|\psi_l\rangle}{\lambda_k+\lambda_l}, & \text{when } \lambda_k+\lambda_l>0 \\ 0, & \text{otherwise} \end{cases}$$

$$F(\rho_\theta) = Tr(\rho_\theta L_\theta^2) = \boxed{2\sum_{\lambda_k+\lambda_l>0} \frac{|\langle\psi_k|\partial_\theta\rho_\theta|\psi_l\rangle|^2}{\lambda_k+\lambda_l}}$$

SUMMARY OF PART 1

* Setting of quantum metrology
* CR Bound
* Fisher Information
* Quantum Fisher Information

PART 2

HEISENBERG LIMIT

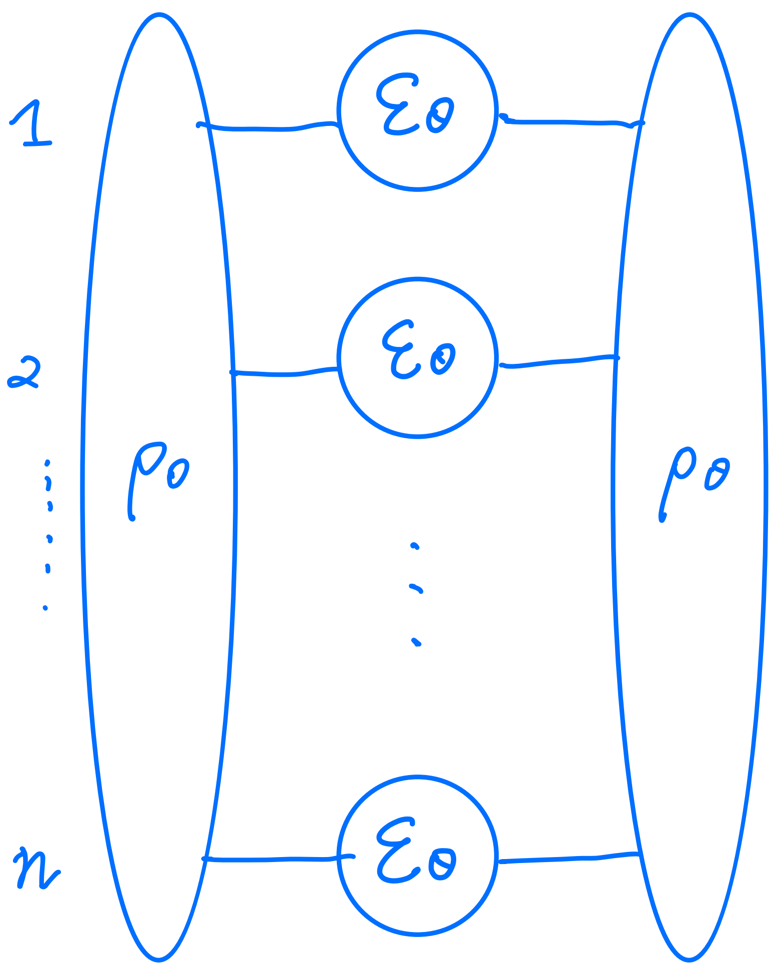
VS STANDARD QUANTUM LIMIT

* QUANTUM CHANNEL ESTIMATION

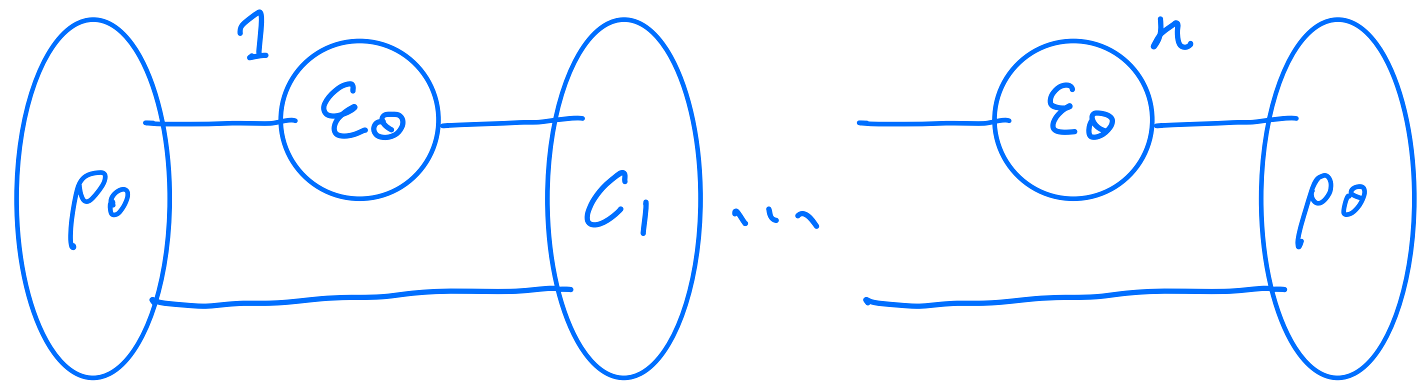
* NOISELESS SCENARIO : PHASE ESTIMATION

* NOISY SCENARIO : PHASE ESTIMATION

QUANTUM CHANNEL ESTIMATION



Parallel



Sequential

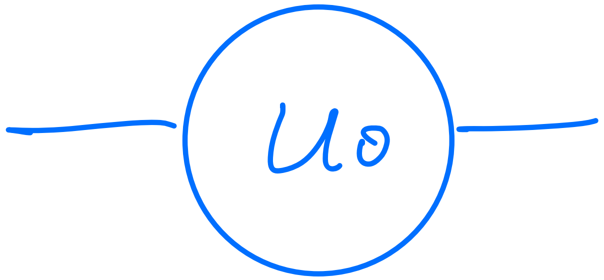
* Heisenberg Limit : $F(\rho_\theta) \sim n^2$

  HL, optimal

* Standard Quantum Limit : $F(\rho_\theta) \sim n$

  SQL, achievable with product state

PHASE ESTIMATION



$$U_\theta = \exp(-i Z \theta), \quad Z = \text{Pauli } Z \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

* Single-qubit case

$$|+\rangle := \frac{|0\rangle + |1\rangle}{\sqrt{2}} \xrightarrow{U_\theta} \frac{e^{-i\theta}|0\rangle + e^{i\theta}|1\rangle}{\sqrt{2}} \cdots \psi_\theta$$

$$\ast \ \{M_\pm\} = \left\{ \frac{|0\rangle \pm |1\rangle}{\sqrt{2}} \frac{\langle 0| \pm \langle 1|}{\sqrt{2}} \right\}$$

$$p(\pm|\theta) = \mathrm{Tr}(M_\pm \psi_\theta) = \cos^2\theta \quad \text{or} \quad \sin^2\theta$$

PHASE ESTIMATION

* FI Calculation:

$$F(p(\pm|\theta)) = \frac{(\partial_\theta \cos^2\theta)^2}{\cos^2\theta} + \frac{(\partial_\theta \sin^2\theta)^2}{\sin^2\theta} = 4(\sin^2\theta + \cos^2\theta) = 4$$

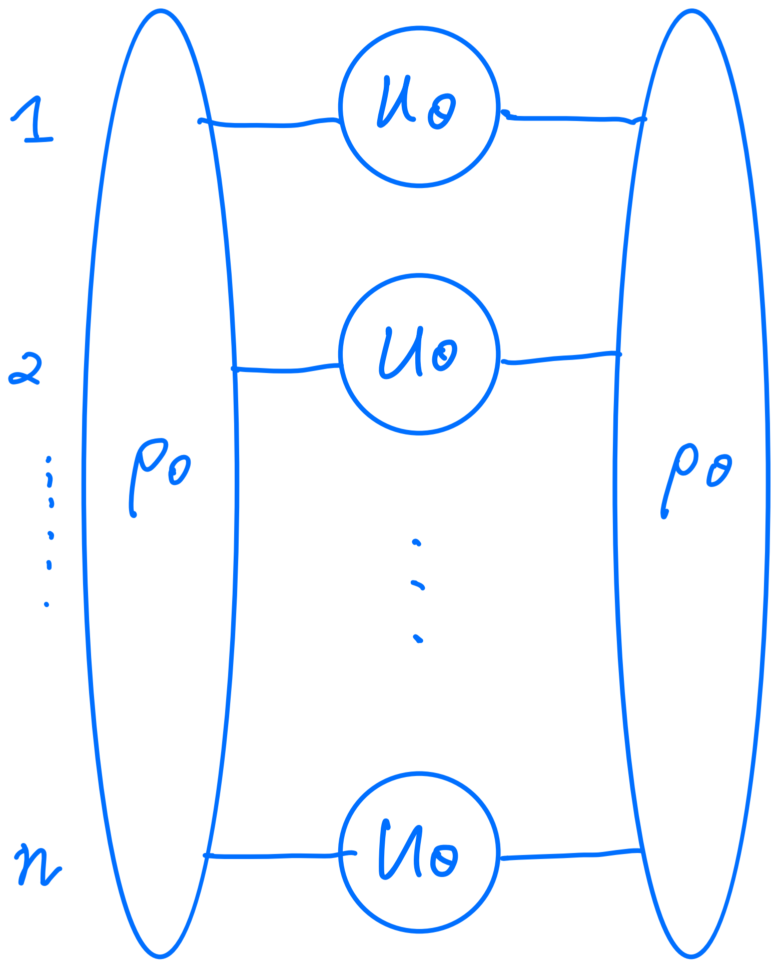
* QFI calculation

$$F(\psi_\theta) = 4\left[\langle\partial_\theta\psi|\partial_\theta\psi\rangle - \langle\psi|\partial_\theta\psi\rangle\langle\partial_\theta\psi|\psi\rangle\right]$$

$$= 4(1-0) = 4$$

* $\{M_\pm\}$ is optimal

PHASE ESTIMATION —— PARALLEL



* n-qubit GHZ state

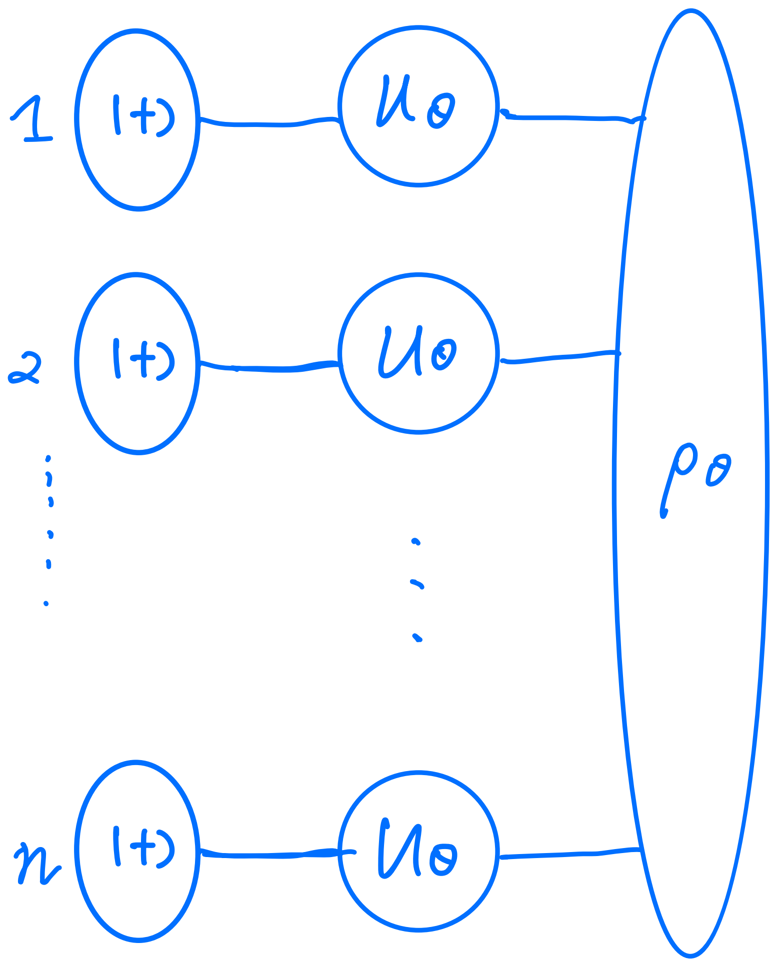
(long-range entanglement)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|0\cdots0\rangle + |1\cdots1\rangle\right) \xrightarrow{u_\theta^{\otimes n}}$$

$$|\psi_\theta\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\theta\cdot n}|0\cdots0\rangle + e^{i\theta\cdot n}|1\cdots1\rangle\right)$$

$$\mathrm{QFI}(\psi_\theta) = 4n^2 \sim \mathrm{ML}$$

PHASE ESTIMATION —— PARALLEL



* n-qubit product state

$$|\psi_0\rangle = |+\rangle^{\otimes n} \xrightarrow{U_\theta^{\otimes n}} \left[\frac{e^{-i\theta}|0\rangle + e^{i\theta}|1\rangle}{\sqrt{2}}\right]^{\otimes n}$$

Using additivity of QFI,

$$\mathrm{QFI}(\psi_\theta) = 4n \quad \sim \text{SQL}$$

PHASE ESTIMATION —— SEQUENTIAL

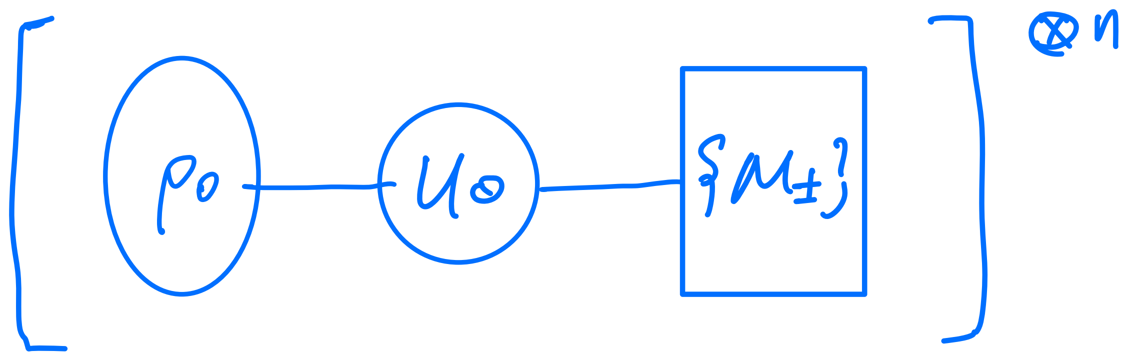
$*$: Input state : $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$

$*$: Output state : $|\psi_\theta\rangle = \left[e^{-in\theta}|0\rangle + e^{in\theta}|1\rangle\right]/\sqrt{2}$

$F(\psi_\theta) = 4n^2$ ~ HL

No control needed.

PHASE ESTIMATION —— SEQUENTIAL



measure n times

* : Input state : $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$

* : Output State : $|\psi_\theta\rangle = \left[e^{-i\theta}|0\rangle + e^{i\theta}|1\rangle\right]/\sqrt{2}$

* : Repeat n times

Total FI : $n \times F(\psi_\theta) = 4n$ $\sim$ SQL

UNITARY ESTIMATION

* $U_\theta = \exp(-iH\theta)$, H: Hermitian operator (Hamiltonian)

* Parallel:

optimal input state : $\frac{1}{\sqrt{2}}\left[|\lambda_{min}\rangle^{\otimes n} + |\lambda_{max}\rangle^{\otimes n}\right]$

$QFI = (\lambda_{max} - \lambda_{min})^2 n^2$ $\sim HL$

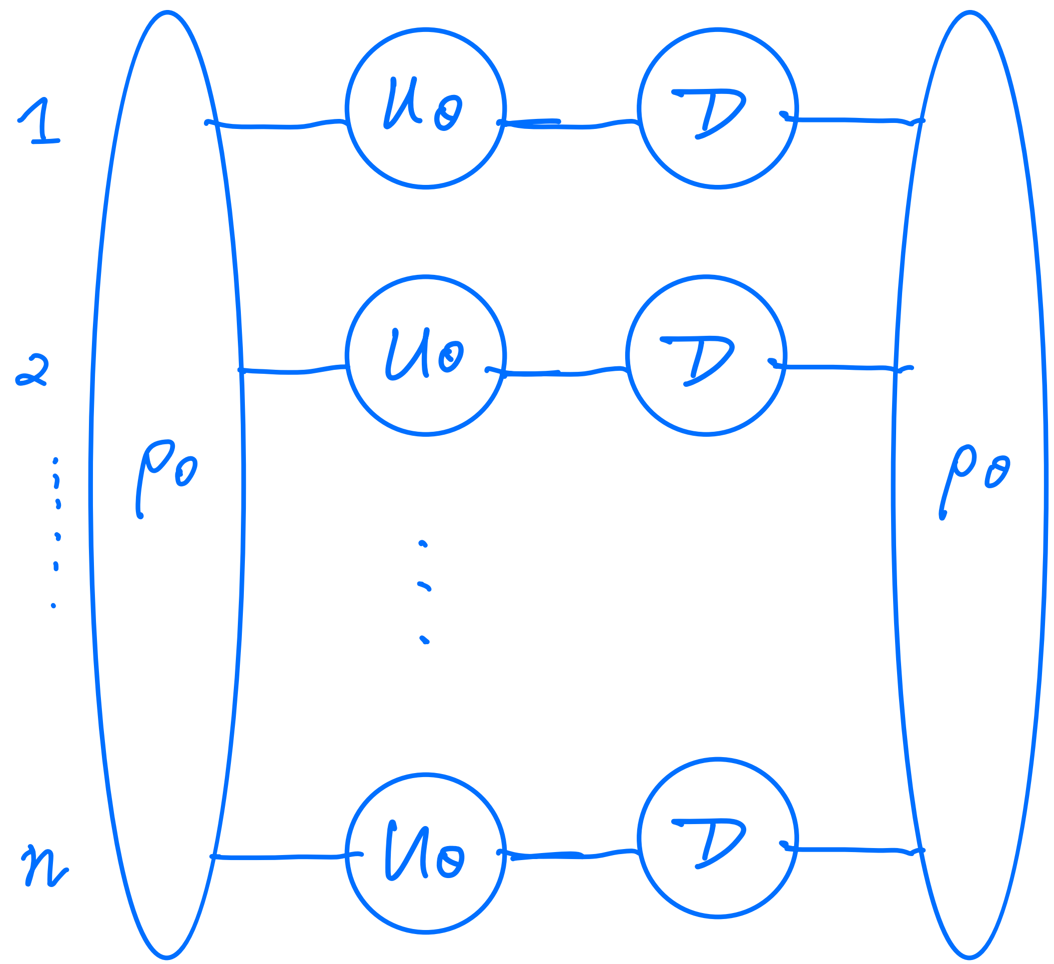
OPTIMAL

* Sequential:

optimal input state : $\frac{1}{\sqrt{2}}\left[|\lambda_{min}\rangle + |\lambda_{max}\rangle\right]$, same QFI

* To achieve HL: long-range entanglement, or long-time coherence

NOISY PHASE ESTIMATION



$\mathcal{D}$: noise channel

Pauli X: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

* Bit-flip noise : $(1-p)(\cdot) + p\, X(\cdot)X$

* Dephasing noise: $(1-p)(\cdot) + p\, Z(\cdot)Z$

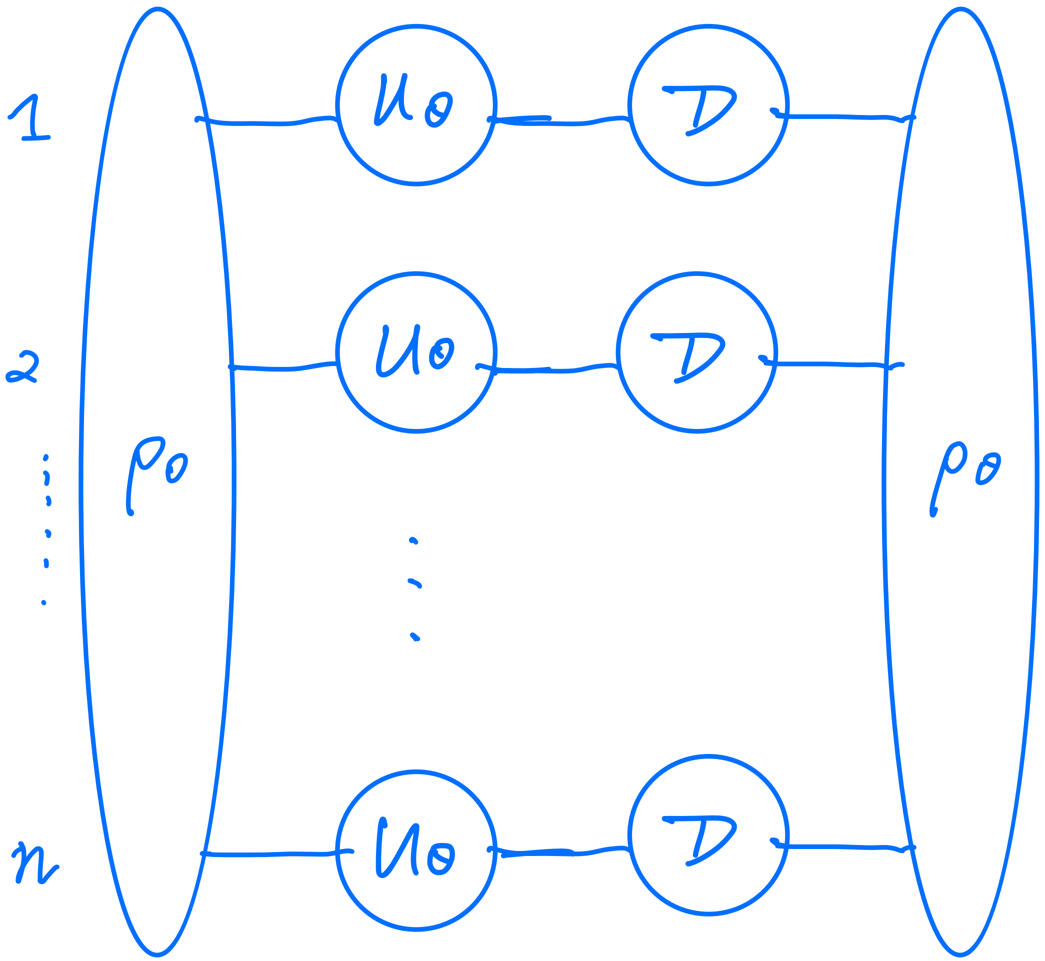
* Depolarizing noise: $(1-p)(\cdot) + p\,(\mathbb{1}/2)$

$$\| $$

$$(1-\tfrac{3}{4}p)(\cdot) + \tfrac{p}{4} X(\cdot)X + \tfrac{p}{4} Y(\cdot)Y + \tfrac{p}{4} Z(\cdot)Z$$

(mixture of X, Y, Z)

NOISY PHASE ESTIMATION —— PARALLEL



* slight decrease in QFI if D acts before U_θ

* n-qubit GHZ state + Bit-flip noise

$$\frac{1}{\sqrt{2}}\left(|0^{\otimes n}\rangle + |1^{\otimes n}\rangle\right) \xrightarrow{U_\theta^{\otimes n}} \frac{1}{\sqrt{2}}\left(e^{-in\theta}|0^{\otimes n}\rangle + e^{in\theta}|1^{\otimes n}\rangle\right)$$

* $\{M_\pm\} = \left\{ \frac{|0\rangle \pm |1\rangle}{\sqrt{2}} \frac{\langle 0| \pm \langle 1|}{\sqrt{2}} \right\}$

Measure parity: (even if product is +1)

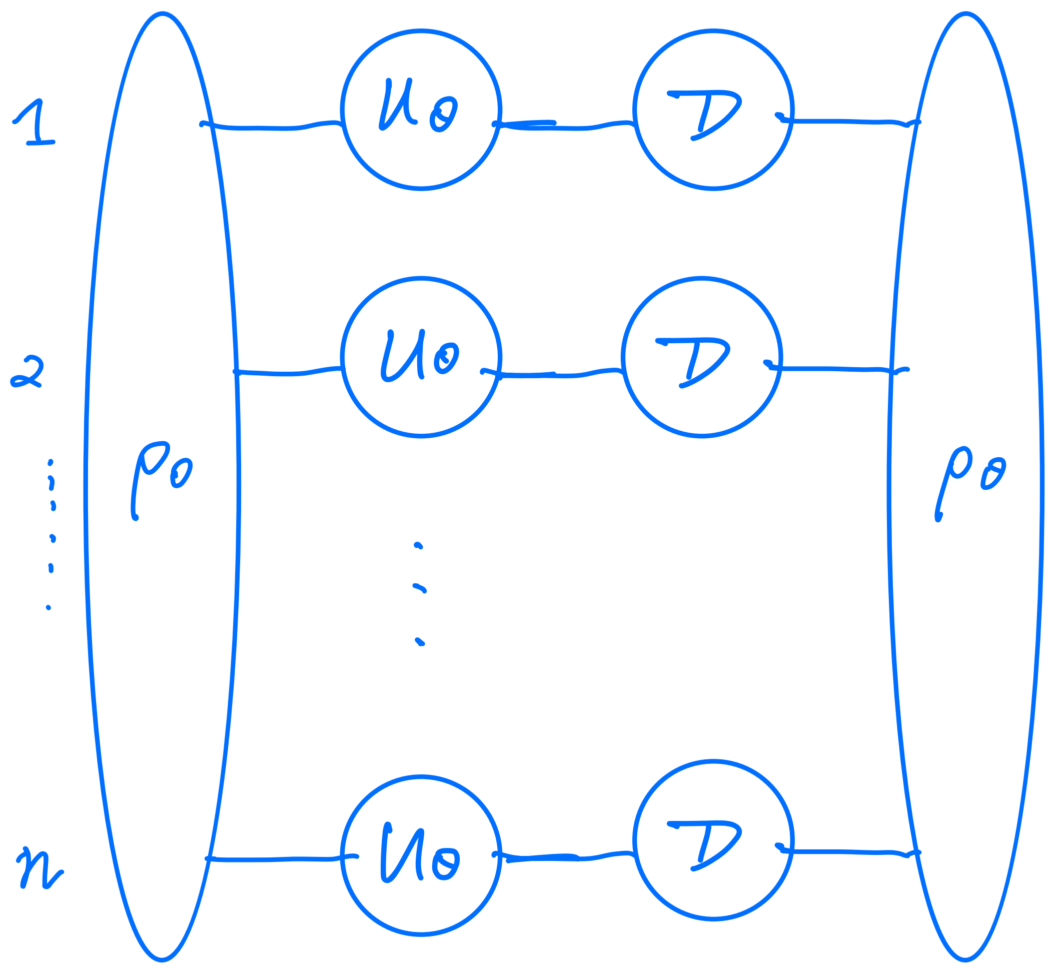
$$p(\text{even}) = \cos^2 n\theta$$

$$p(\text{odd}) = \sin^2 n\theta$$

* $\{M_\pm\}$ outcome invariant under bit-flip

$\Rightarrow$ QFI $= 4n^2$ ~ the same as noiseless case

NOISY PHASE ESTIMATION —— PARALLEL



* n-qubit GHZ state + Dephasing noise

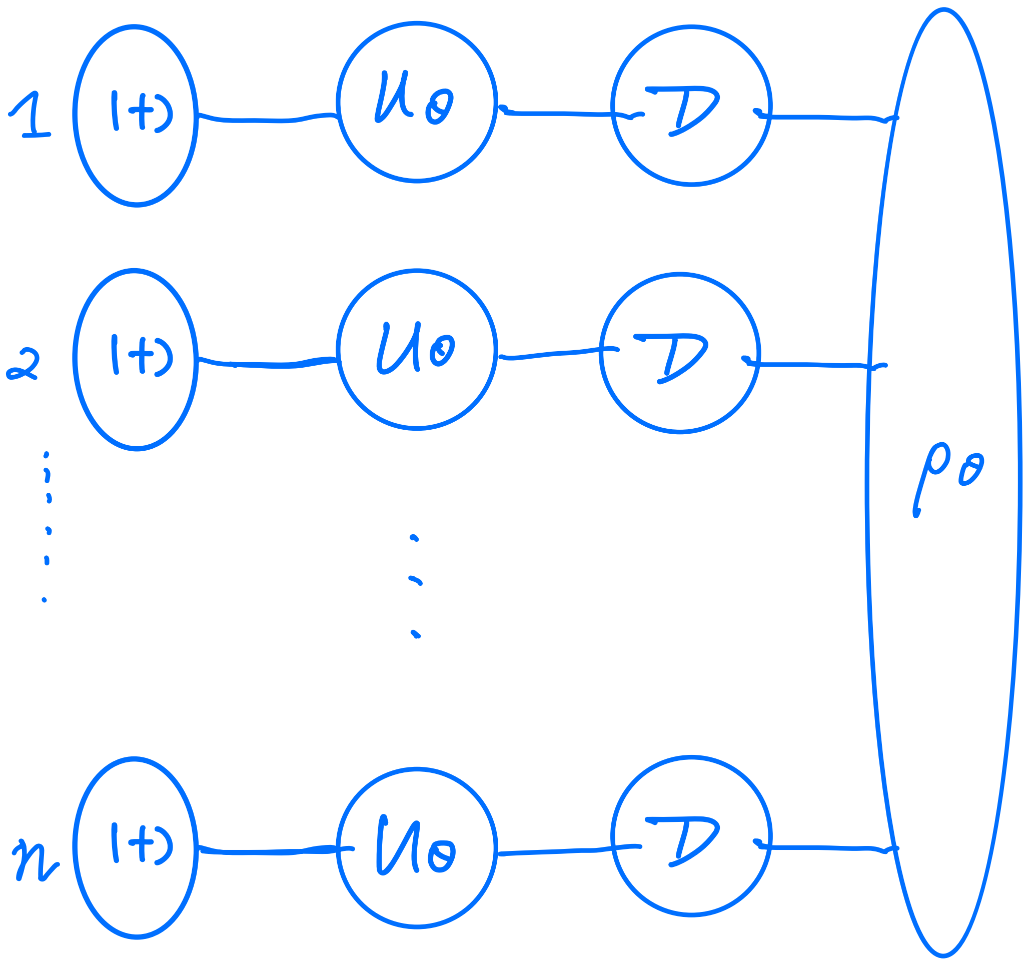
$$\frac{1}{\sqrt{2}}\left(|0^{\otimes n}\rangle + |1^{\otimes n}\rangle\right) \xrightarrow{\mathcal{U}_\theta^{\otimes n}} \frac{1}{\sqrt{2}}\left(e^{-in\theta}|0^{\otimes n}\rangle + e^{in\theta}|1^{\otimes n}\rangle\right)$$

$$\xrightarrow{\mathcal{D}^{\otimes n}} \frac{1}{2}\begin{bmatrix} 1 & (1-2p)^n e^{-i2n\theta} \\ (1-2p)^n e^{i2n\theta} & 1 \end{bmatrix}$$

$$|\vec{0}\rangle + |\vec{1}\rangle \xrightarrow{Z^{\vec{s}} = Z^{s_1}\otimes Z^{s_2}\cdots\otimes Z^{s_n}} \underbrace{|\vec{0}\rangle + (-1)^{|s|}|\vec{1}\rangle}_{\text{two-level}}$$

$$\text{QFI} = 4n^2(1-2p)^{2n}$$ — exponential decay

NOISY PHASE ESTIMATION —— PARALLEL

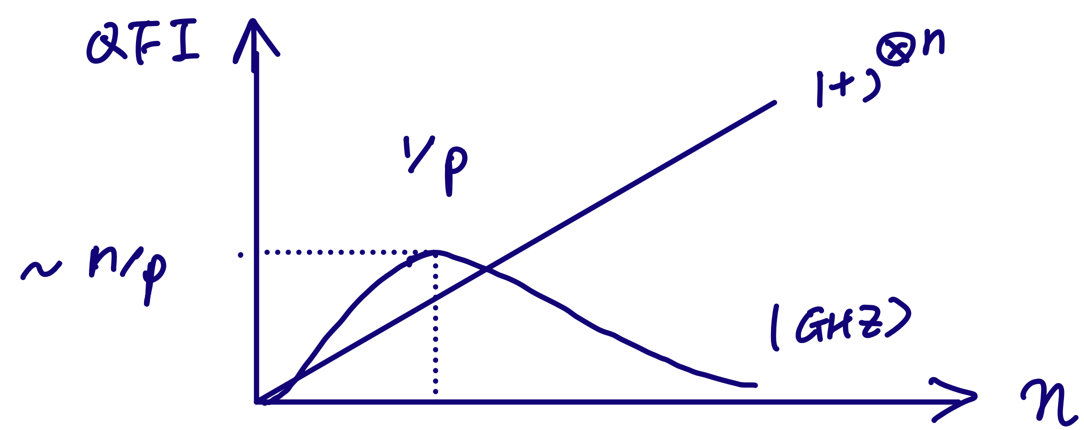


* n-qubit product state + Dephasing noise

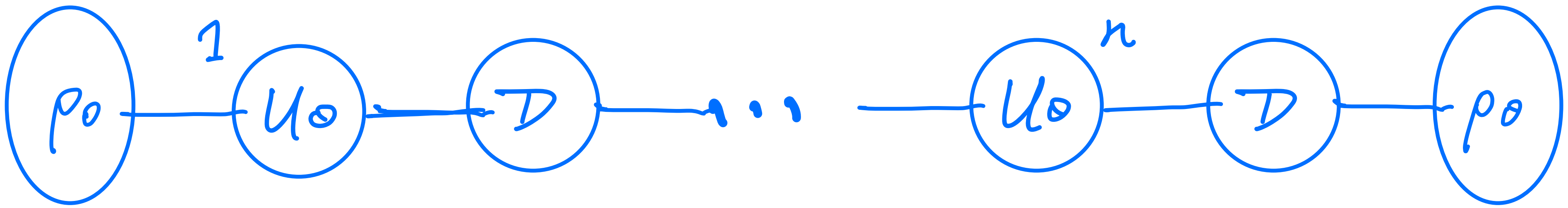
$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \xrightarrow{U_\theta} \frac{1}{\sqrt{2}}\left(e^{-i\theta}|0\rangle + e^{i\theta}|1\rangle\right)$$

$$\xrightarrow{D} \frac{1}{2}\begin{bmatrix} 1 & (1-2p)\,e^{-i2\theta} \\ (1-2p)\,e^{i2\theta} & 1 \end{bmatrix}$$

$$\mathrm{QFI} = 4n\,(1-2p)^2 \sim \text{SQL}$$



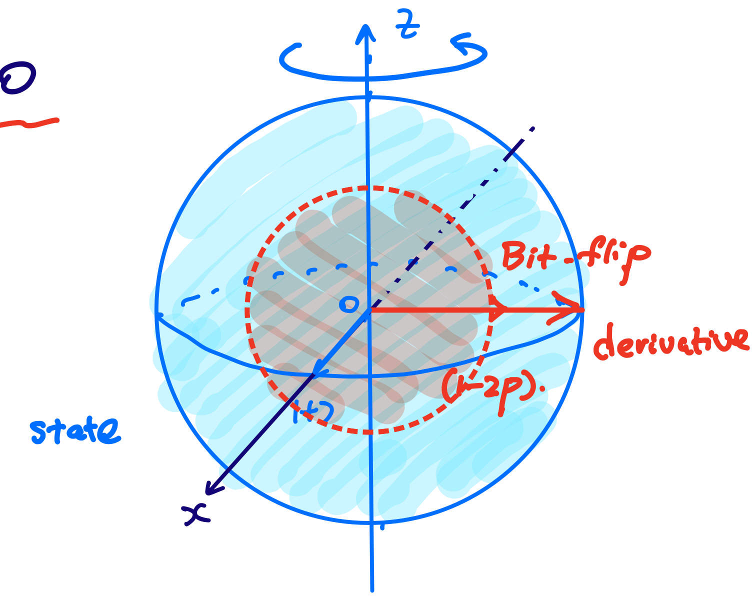
NOISY PHASE ESTIMATION —— SEQUENTIAL



* : Bit-flip noise, calculate QFI at $\theta = 0$

* : Input state : $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$

* : Output State : Bloch Representation



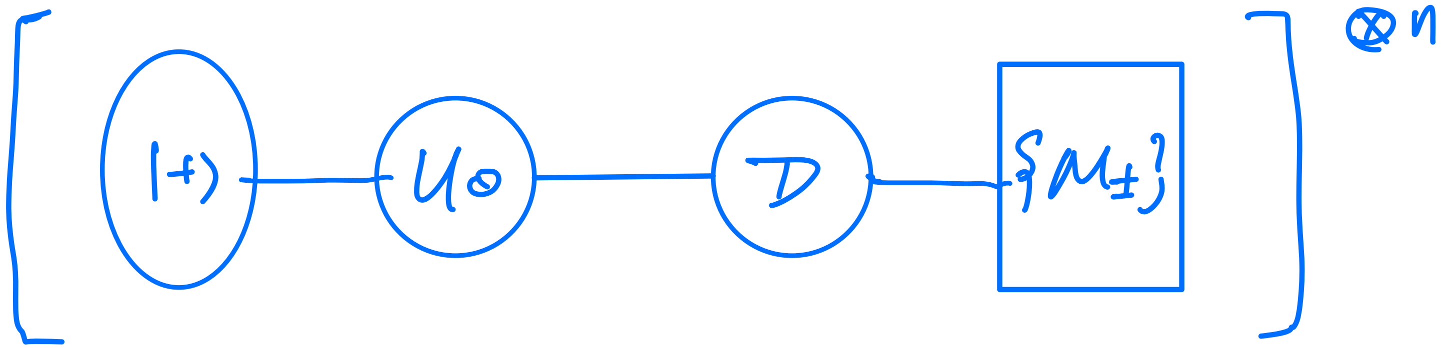
$$(1,0,0) \xrightarrow{D} \cdots \xrightarrow{D} (1,0,0)$$

derivative w.r.t. θ

$$(1,0,0) \xrightarrow{\partial_\theta U_\theta} (0,1,0) \xrightarrow{D} (0,\ 1-2p,\ 0) \longrightarrow \cdots \xrightarrow{D} (0,\ (1-2p)^n,\ 0)$$

$$\mathrm{QFI} = 4\left[(1-2p) + \cdots + (1-2p)^n\right]^2$$ ---- approaching a constant

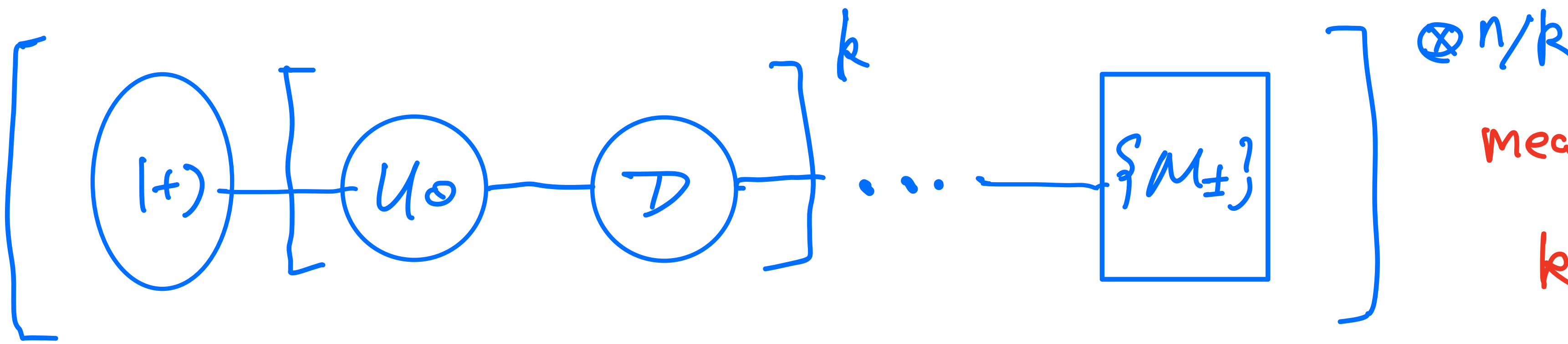
NOISY PHASE ESTIMATION — SEQUENTIAL



Bit-flip

measure n times

$$\text{QFI} = 4(1-2p)^2 n \quad \sim \text{SQL}$$



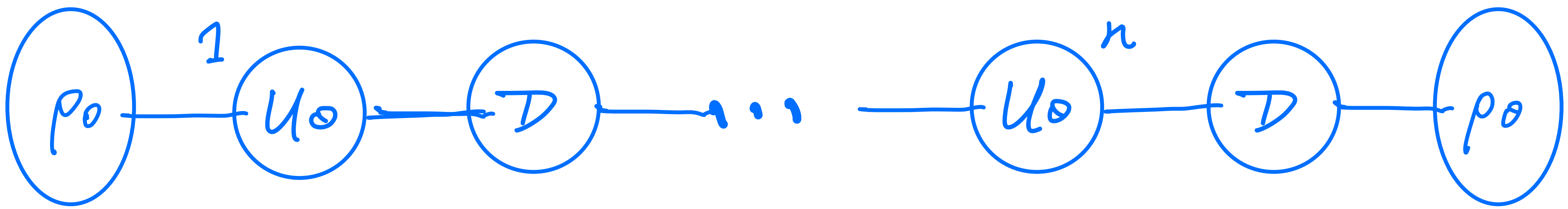
Bit-flip

measure n/k times

k is optimal at $\sim 0.6/p$

$$\text{QFI} \approx 0.8\, n/p \quad \sim \text{SQL}$$

NOISY PHASE ESTIMATION —— SEQUENTIAL



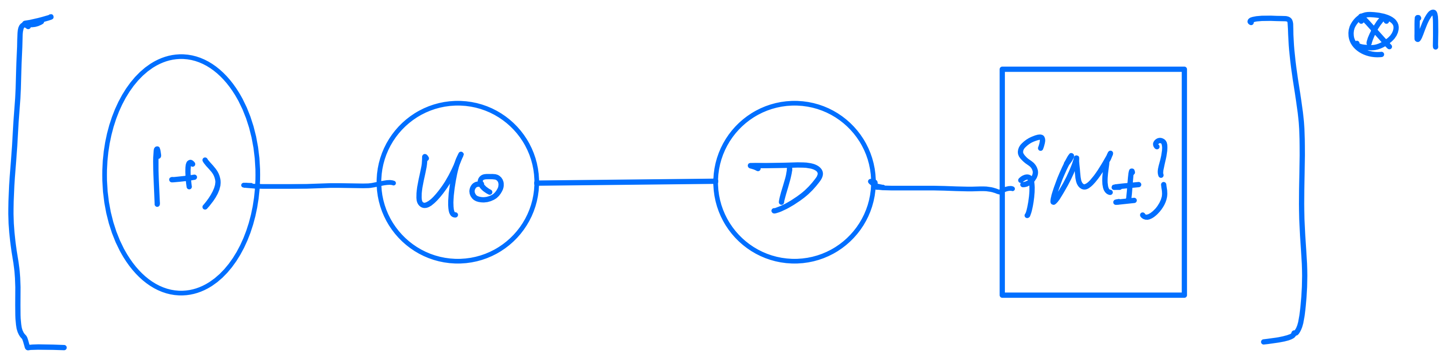
* : Dephasing noise

* : Input state : $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$

* : Output State : $\rho_\theta = \frac{1}{2}\begin{bmatrix} 1 & (1-2p)^n e^{-2in\theta} \\ (1-2p)^n e^{2in\theta} & 1 \end{bmatrix}$

$\text{QFI} = 4n^2(1-2p)^{2n}$ ---- decays exponentially

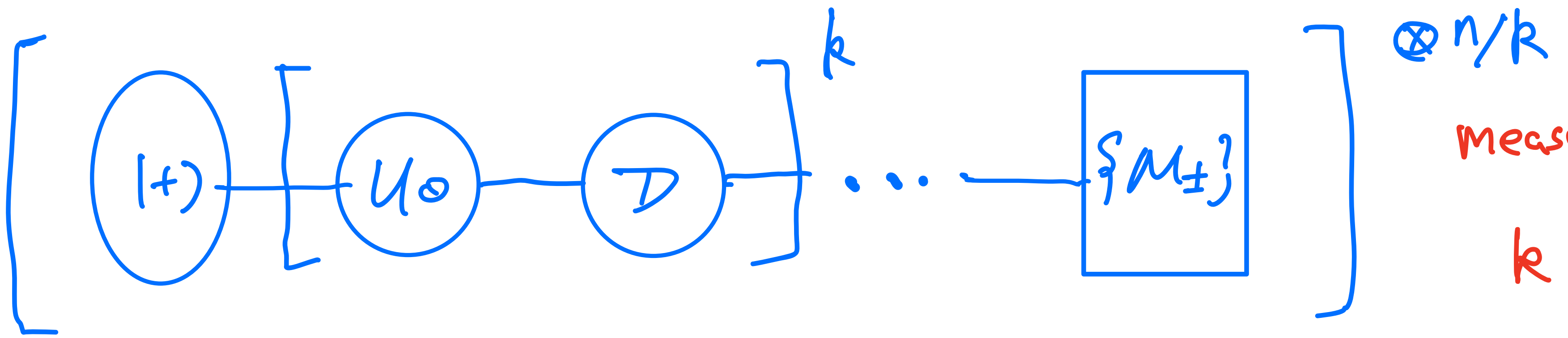
NOISY PHASE ESTIMATION — SEQUENTIAL



Dephasing

measure n times

$$QFI = 4(1-2p)^2 n \quad \sim SQL$$



Dephasing

measure n/k times

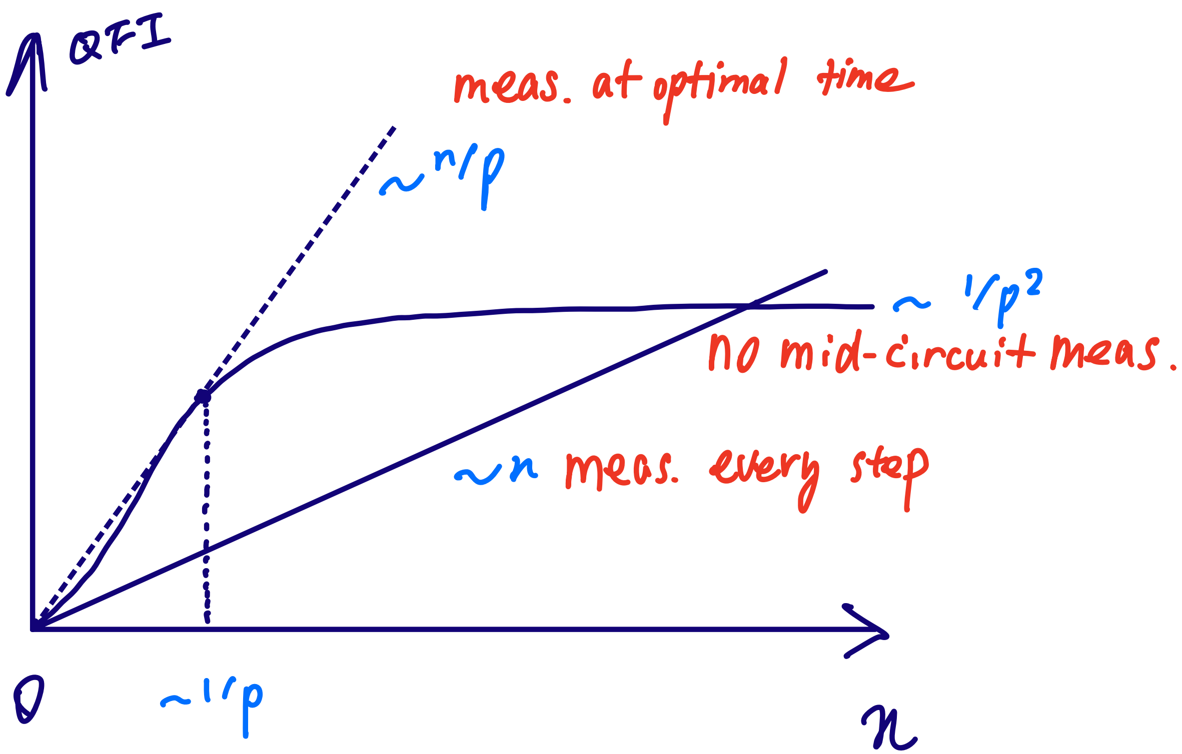
k is optimal at $\sim 0.25/p$

$$QFI \approx 0.37\, n/p \quad \sim SQL$$

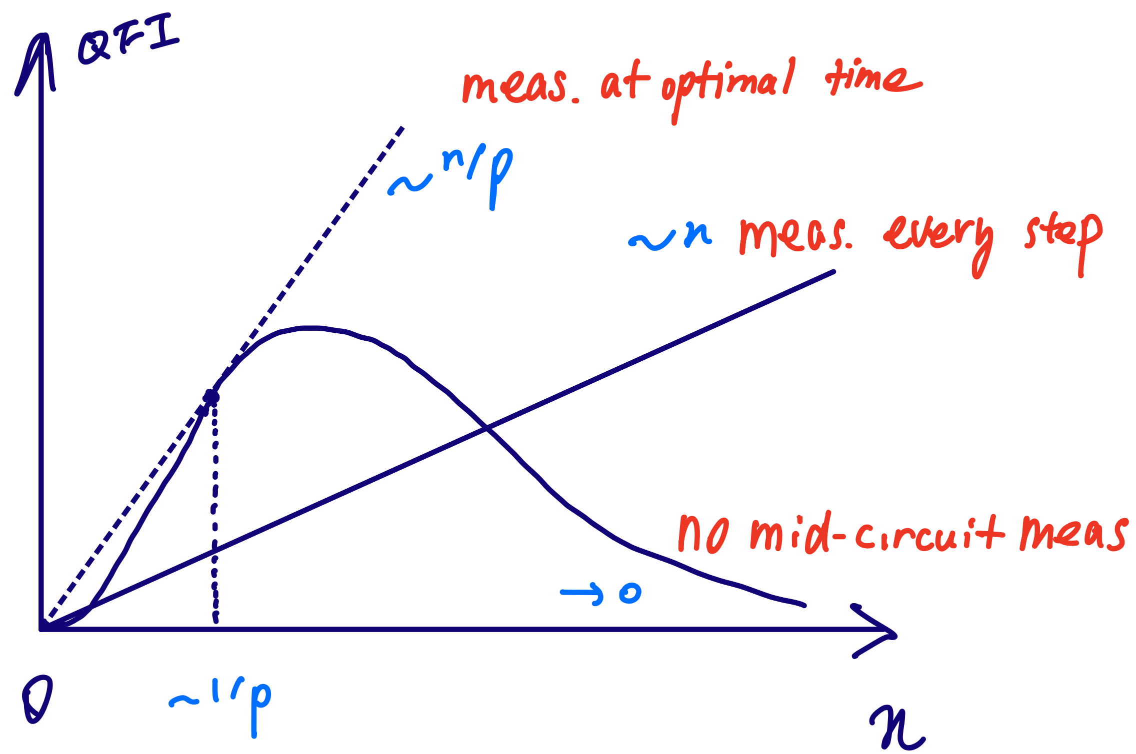
NOISY PHASE ESTIMATION — SEQUENTIAL

* input state: $|+\rangle$, with noise

BIT-FLIP — SQL



DEPHASING — SQL



SUMMARY OF PART 2

* Definitions of HL, SQL, Parallel, Sequential
* Phase estimation, Achieving the HL, GHZ state
* Quantum Noise, Bit-flip, Dephasing noises
* Noisy phase estimation, Achieving the HL in parallel strategy
* In most cases, FI follows the SQL

Part 3

INTRODUCTION TO QUANTUM ERROR CORRECTION

* QUANTUM NOISE

* QUANTUM ERROR CORRECTION CONDITIONS

* Examples:
  - Repetition Code
  - Shor Code
  - Surface code

QUANTUM NOISE

* $\mathcal{D}(\rho) = \sum_i K_i (\cdot) K_i^{\dagger}$, K_i: Kraus operators

  s.t. $\sum_i K_i^{\dagger} K_i = \mathbb{1}$

  CPTP map: $\mathrm{Tr}(\mathcal{D}(\rho)) = \mathrm{Tr}(\rho)$

  $(\mathcal{D} \otimes \mathbb{1})(\rho) \geq 0$, if $\rho \geq 0$

* K_i : noise operators

  i : indices of different types of noise

* Kraus representation is not unique $\begin{bmatrix} K_1' \\ \vdots \\ K_{r'}' \end{bmatrix} = U \begin{bmatrix} K_1 \\ \vdots \\ K_r \end{bmatrix}$

  $U^{\dagger} U = \mathbb{1}_r$

QUANTUM NOISE

* Dephasing :

$$K_0 = \sqrt{1-p}\ \mathbb{1}$$
$$K_1 = \sqrt{p}\ Z$$

* Bit-flip :

$$K_0 = \sqrt{1-p}\ \mathbb{1}$$
$$K_1 = \sqrt{p}\ X$$

* Depolarizing :

$$K_0 = \sqrt{1-3p/4}\ \mathbb{1}$$
$$K_1 = \sqrt{p/4}\ X$$
$$K_2 = \sqrt{p/4}\ Y$$
$$K_3 = \sqrt{p/4}\ Z$$

QUANTUM ERROR CORRECTION CONDITION

* QUESTION: P is the projector onto code subspace,
"logical space"

Under what conditions, $\exists\, R$, decoding channel,

$$\underbrace{R}_{\text{perfect recovery}} \circ D(\rho) = \rho \, , \text{ for any } \underbrace{\rho = P\rho P}_{\rho \text{ is in the code subspace}}$$

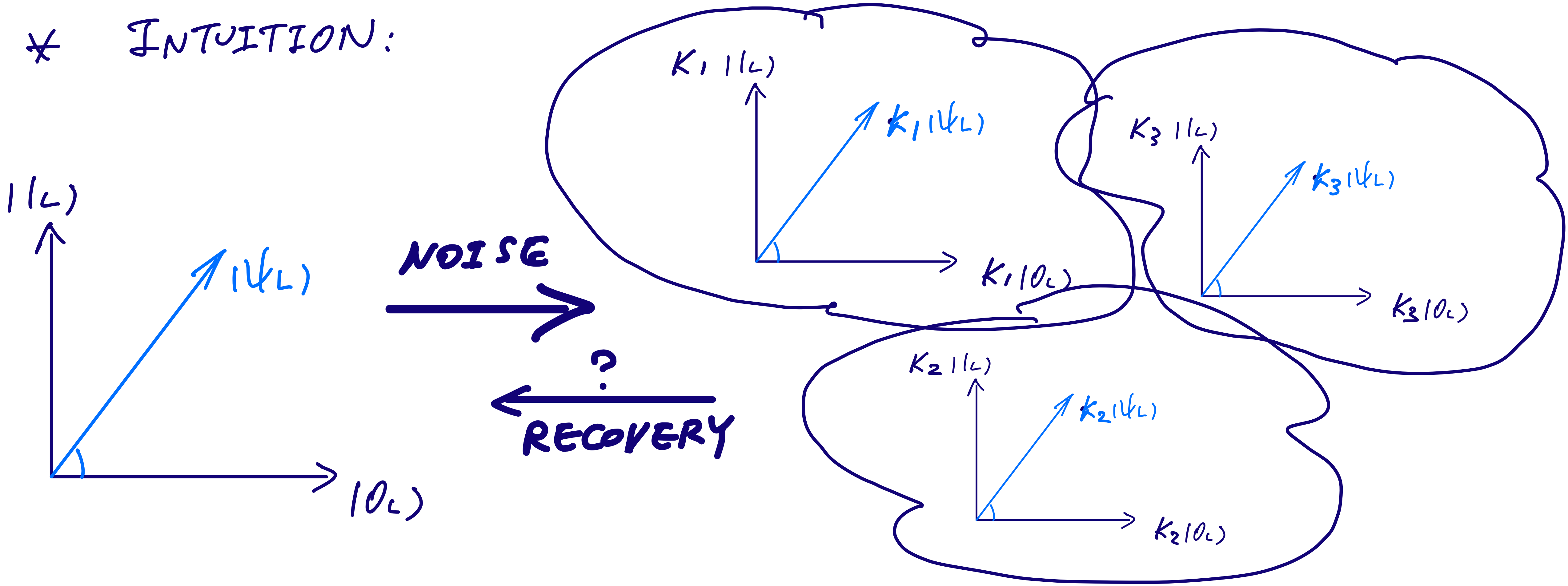
* ANSWER: The necessary & sufficient condition:

$$\boxed{P K_i^{\dagger} K_j P = \alpha_{ij} P \, , \quad \forall\, i,j, \text{ some } \alpha_{ij} \in \mathbb{C}}$$

Knill - Laflamme Condition

QUANTUM ERROR CORRECTION CONDITION

* INTUITION:



$\langle 0_L | K^\dagger \; K | 1_L \rangle = 0$ (orthogonal)

$\langle 0_L | K^\dagger K | 0_L \rangle = \langle 1_L | K^\dagger K | 1_L \rangle$ (same scale)

$$\Longrightarrow \boxed{P K^\dagger K P \propto P}$$

QUANTUM ERROR CORRECTION CONDITION

* SUFFICIENCY

$$\boxed{P K_i^\dagger K_j P = \alpha_{ij} P, \quad \forall i,j, \quad \text{some } \alpha_{ij} \in \mathbb{C}}$$

$\alpha = u \lambda u^\dagger$, u is unitary, λ is diagonal

$\vec{F} = u^T \vec{K}$ is another Kraus representation $P F_i^\dagger F_j P = \lambda_{ij} \delta_{ij} P$

$F_k P = \sqrt{\lambda_k}\, U_k P$ for some U_k, using polar decomposition.

$P_k = U_k P U_k^\dagger$, $\quad R(\cdot) = \boxed{\sum_k U_k^\dagger P_k (\cdot) P_k U_k}$

Detection P_k

Correction U_k

Satisfies $R \circ D(\rho) = \rho$. for $P \rho P = \rho$.

QUANTUM ERROR CORRECTION CONDITION

* NECESSITY

$$\mathcal{R}\circ\mathcal{D}(P\rho P) = P\rho P, \quad \forall \rho$$

$\{R_j K_i P\}$ and $\{P\}$ are Kraus operators of the same map,

$\Rightarrow$ $R_j K_i P = c_{ji} P$ for some c_{ji}

$$\Rightarrow \quad P K_i^\dagger K_j P = P K_i^\dagger \overbrace{\left[\sum_k R_k^\dagger R_k\right]}^{=\mathbb{I}} K_j P = \sum_k c_{ki}^* c_{kj} P = \alpha_{ij} P.$$

STABILIZER CODE

* n-qubit Pauli group : $\{\mathbb{1}, X, Y, Z\}^{\otimes n} \times \{\pm 1, \pm i\}$

* Stabilizer group $\mathcal{S}$: subgroup of Pauli group satisfying
  (1) $[s_i, s_j] = 0$, (2) $-\mathbb{1} \notin \mathcal{S}$

* Stabilizer generators : $\mathcal{S} = \langle g_1, g_2, \cdots g_{n-k} \rangle$, $k \geq 1$

* Logical space : $\{ |\psi\rangle : s|\psi\rangle = |\psi\rangle \ \forall s \in \mathcal{S} \}$ contains k logical qubits

* $$P = \prod_{i=1}^{n-k} \frac{\mathbb{1} + g_i}{2} = \frac{1}{2^{n-k}} \sum_{s \in \mathcal{S}} s$$

Sanity check : $$SP = \frac{1}{2^{n-k}} \underbrace{\sum_{s' \in \mathcal{S}}}_{} ss' = \frac{1}{2^{n-k}} \underbrace{\sum_{ss' \in \mathcal{S}}}_{} ss' = P$$

$\mathcal{S}$ is a group

STABILIZER CODE

* Consider Pauli noise $\mathcal{D}(\cdot) = \sum_i K_i (\cdot) K_i^\dagger$
  where each K_i is proportional to a Pauli operator E_i

* QEC condition: $P E_i^\dagger E_j P \propto P$

✓ Situation 1: $E_i^\dagger E_j$ anti-commutes with some stabilizer g

$$P E_i^\dagger E_j g P = -P g E_i^\dagger E_j P \;,\quad gP = Pg = P$$

$$\Rightarrow \quad P E_i^\dagger E_j P = -P E_i^\dagger E_j P \Rightarrow P E_i^\dagger E_j P = 0$$

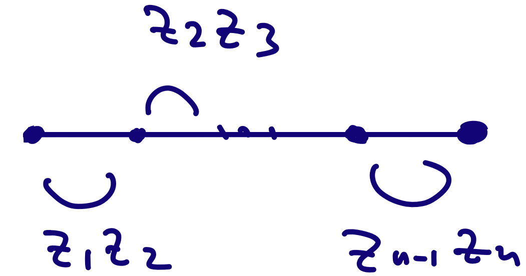
✓ Situation 2: $E_i^\dagger E_j \propto$ a stabilizer $\Longrightarrow P E_i^\dagger E_j P \propto P$

× Situation 3: $E_i^\dagger E_j$ commutes with all stablizers, but is not proportional to one

If $P E_i^\dagger E_j P = \alpha P$, then $E_i^\dagger E_j |\psi_L\rangle = \alpha |\psi_L\rangle$ for all $|\psi_L\rangle$. $E_i^\dagger E_j / \alpha$ must be a stabilizer, Contradiction.

$E_i^\dagger E_j$ is called LOGICAL operators in situation 3.

STABILIZER CODE



* Example : Repetition Code

Stabilizers: $\langle Z_1Z_2, Z_2Z_3, \cdots Z_{n-1}Z_n \rangle = \langle Z_iZ_j \rangle$

Codewords: $|00\cdots 0\rangle$, $|1\cdots 1\rangle$

Errors: $X^{t_1} \otimes \cdots \otimes X^{t_n}$, $|t| \leq \lfloor (n-1)/2 \rfloor$

why : $E_1 = X^t$, $E_2 = X^s$, $|t \oplus s| \leq 2\lfloor (n-1)/2 \rfloor < n$

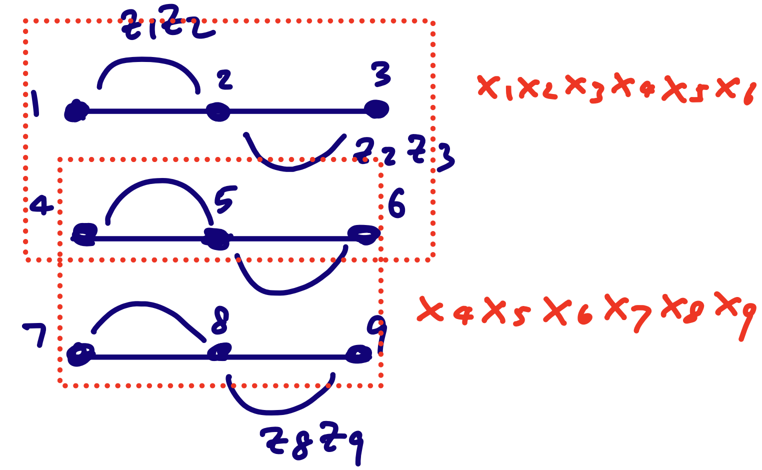
✓ $E_1^\dagger E_2$ anticommutes with Z_iZ_j, $i \in t \cup s$, $j \notin t \cup s$

✗ $E_1 = Z_1$, $E_2 = \mathbb{1}$, $E_1^\dagger E_2 = Z_1$, Commutes with $\mathcal{S}$ but is not inside

Logical operator: $Z_L = Z_i$, $X_L = X_1X_2\cdots X_n$

Decoding : Majority voting

STABILIZER CODE



* Example : Shor code

Stabilizers: $Z_1Z_2,\ Z_2Z_3,\ Z_4Z_5,\ Z_5Z_6,\ Z_7Z_8,\ Z_8Z_9$

$X_1X_2X_3X_4X_5X_6,\ X_4X_5X_6X_7X_8X_9$

Codewords: $$\left(\frac{|000\rangle \pm |111\rangle}{\sqrt{2}}\right)^{\otimes 3} = \begin{cases} |+_R +_R +_R\rangle \\ |-_R -_R -_R\rangle \end{cases}$$

Errors: Single-qubit Errors

why : X noise, corrected by Z stabs

Z noise, corrected by X stabs.

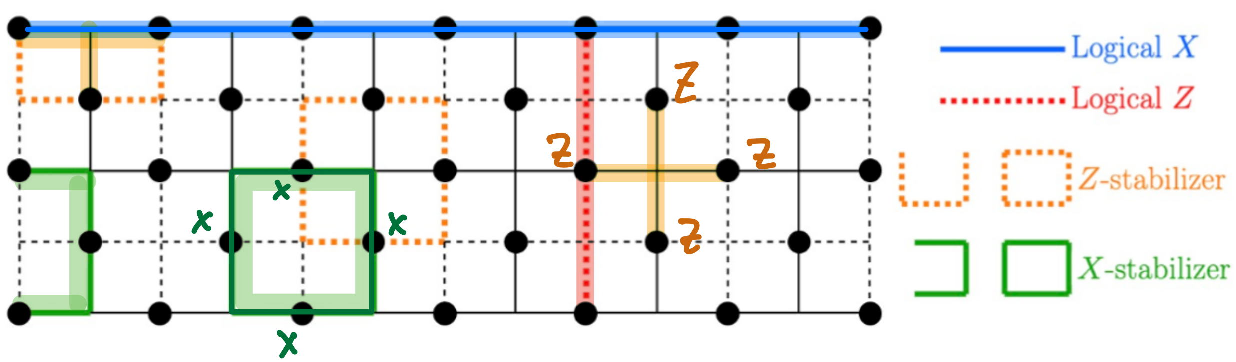
Logical operation: $Z_L = X_1X_2X_3X_4X_5X_6X_7X_8X_9$

$X_L = Z_1Z_4Z_7$

STABILIZER CODE

* Example : (Thin) Surface code

Stabilizers:



Errors: Single-qubit Errors

why: X noise, corrected by Z stabs

Z noise, corrected by X stabs.

SUMMARY OF PART 3

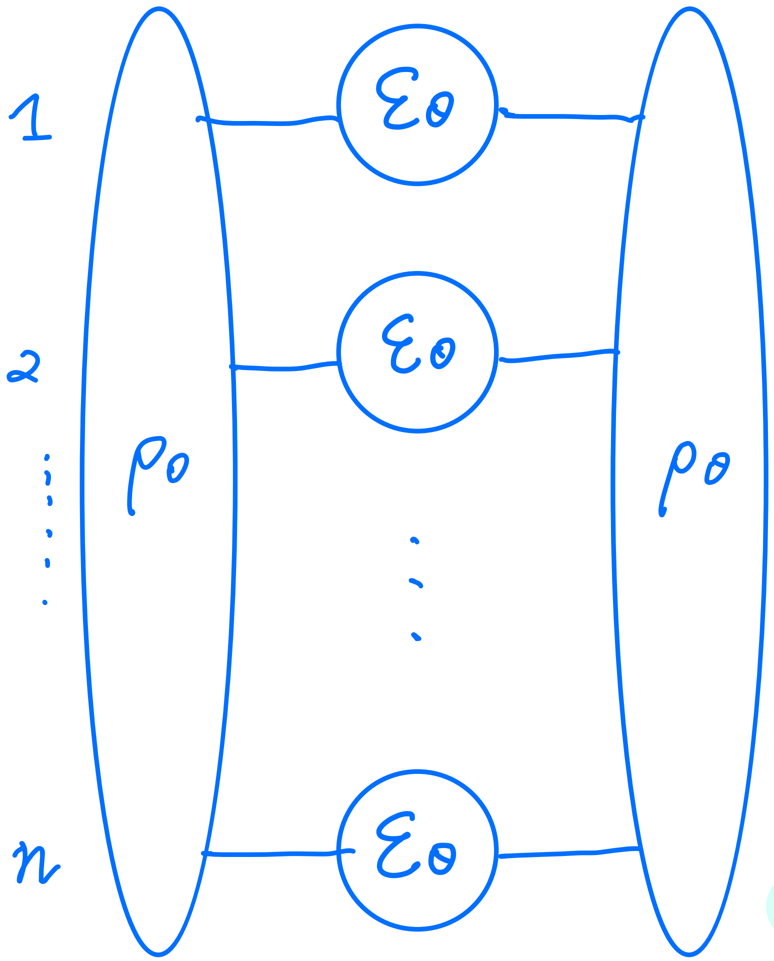
* Definitions Quantum Noise (channel), Kraus Representation
* QEC Condition ; Necessary and Sufficient
  (KL)
* Stabilizer Code ; QEC Condition
* Example : Repetition code
  Shor code (encoding one logical qubit)
  Surface code

Part 4

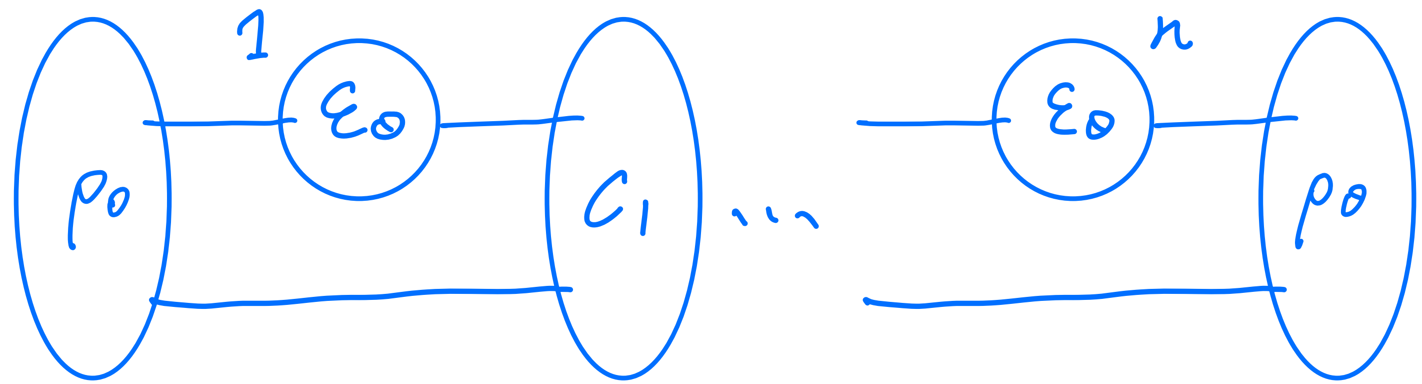
QUANTUM ERROR CORRECTION FOR METROLOGY

* NOISY QUANTUM CHANNEL ESTIMATION
* HAMILTONIAN-NOT-IN-KRAUS-SPAN CONDITION
* EXAMPLE: PHASE ESTIMATION UNDER BIT-FLIP NOISE
* EXAMPLE: ZZZ ESTIMATION UNDER SINGLE-QUBIT NOISE

QUANTUM CHANNEL ESTIMATION



Parallel

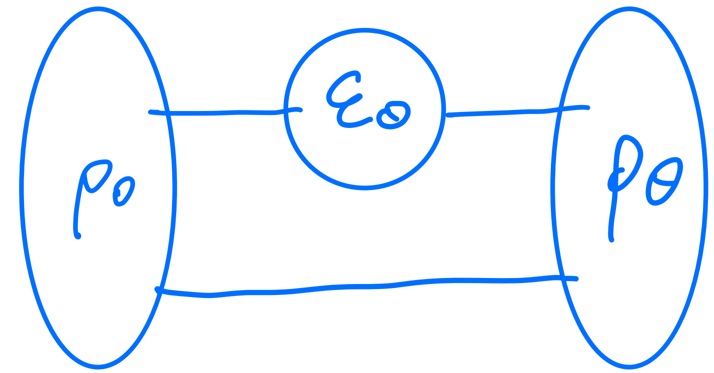


Sequential

$$\max_{\substack{\text{all par(seq)} \\ \text{protocols}}} F^{(\text{par, seq})}(\rho_\theta) = \begin{cases} \sim n^2 & \text{HL} \\ \sim n & \text{SQL} \end{cases}$$

SINGLE-CHANNEL ESTIMATION

* $\mathcal{E}_\theta(\rho) = \sum_i K_{i,\theta}(\rho)\, K_{i,\theta}^\dagger$



* $F(\mathcal{E}_\theta) := \max_{\rho_0} F\big((\mathcal{E}_\theta \otimes \mathbb{1})(\rho_0)\big)$

CHANNEL QFI

* $F(\mathcal{E}_\theta) = 4 \min_{u_\theta :\, u_\theta^\dagger u_\theta = \mathbb{1}} \left\| \partial_\theta \vec{K'_\theta}^\dagger \cdot \partial_\theta \vec{K'_\theta} \right\| = \boxed{4 \min_h \|\alpha\|}$

$$\vec{K'_\theta} = \begin{bmatrix} K'_{1,\theta} \\ \vdots \\ K'_{r',\theta} \end{bmatrix} = u_\theta \begin{bmatrix} K_{1,\theta} \\ \vdots \\ K_{r,\theta} \end{bmatrix}, \quad h = i\, u_\theta^\dagger (\partial_\theta u_\theta), \quad \alpha = \left(\partial_\theta \vec{K_\theta} - i h \vec{K_\theta}\right)^\dagger \left(\partial_\theta \vec{K_\theta} - i h \vec{K_\theta}\right)$$

minimize over all possible Kraus representation $\Longleftrightarrow$ minimize over all Hermitian h

MULTI-CHANNEL ESTIMATION

$$* \quad F_n^{(par)} \leq 4 \min_h \left(\|\alpha\| \overset{\text{SQL}}{n} + \|\beta\|^2 \overset{\text{HL}}{n(n-1)} \right)$$

$$* \quad F_n^{(seq)} \leq 4 \min_h \left(\|\alpha\| n + \|\beta\| \left(\|\beta\| + 2\sqrt{\|\alpha\|} \right) n(n-1) \right)$$

$$\begin{cases} \alpha = \left(\partial_\theta \vec{K}_\theta - i h \vec{K}_\theta\right)^\dagger \left(\partial_\theta \vec{K}_\theta - i h \vec{K}_\theta\right) \\ \beta = i \vec{K}_\theta^\dagger \left(\partial_\theta \vec{K}_\theta - i h \vec{K}_\theta\right) = i \vec{K}_\theta^\dagger\left(\partial_\theta \vec{K}_\theta\right) + \vec{K}_\theta^\dagger h \vec{K}_\theta =: \underbrace{H_\theta}_{\text{Hamiltonian}} + \vec{K}_\theta^\dagger h \vec{K}_\theta \end{cases}$$

$$* \quad \boxed{\exists h, \quad \beta = 0} \implies F_n^{(par)} \leq F_n^{(seq)} \leq O(n)$$

$$\Updownarrow$$

$$\boxed{H_\theta \in \operatorname{span}\{ K_{i,\theta}^\dagger K_{j,\theta}, \ \forall i,j \}}$$

"Hamiltonian-in-Kraus-span"

THEOREM STATEMENT

* $\exists$ parallel (or sequential strategy). s.t.

$$F^{(par)} \; (\text{or } F^{(seq)}) = \Theta(n^2)$$

IF AND ONLY IF

the HNKS condition is satisfied.

$$\boxed{H_\theta \notin \operatorname{span}\{ K_{i,\theta}^\dagger K_{j,\theta},\ \forall i,j \}}$$

"Hamiltonian-not-in-Kraus-span"
HNKS Condition

* If HNKS violated, $F^{(par,seq)} = O(n)$, (SQL)

SUFFICIENCY: QEC PROTOCOL.

* QEC condition: $P K_{i,\theta}^\dagger K_{j,\theta} P \propto P \quad \forall i, j$

* <u>Goal</u>: Find a QEC code P, s.t. $\boxed{\begin{cases} P K_{i,\theta}^\dagger K_{j,\theta} P \propto P \\ P H_\theta P \not\propto P \end{cases}}$

$\exists$ Recovery R, s.t.

$$\rho_{\theta+d\theta} = R \circ \mathcal{E}_{\theta+d\theta}(\rho) = \rho_\theta - i[\underbrace{P H_\theta P}_{H_c}, \rho_\theta] d\theta + O(d\theta^2)$$

Assume $P K_i^\dagger K_j P = \lambda_{ij} \delta_{ij} P$ (which can be achieved changing the Kraus rep)

Then let $K_i P = \sqrt{\lambda_i} U_i P$. $P_i = U_i P U_i^\dagger$, $R = \sum_i U_i^\dagger P_i (\cdot) P_i U_i = \sum_i P U_i^\dagger (\cdot) U_i P$

$$\mathcal{E}_{\theta+d\theta}(\rho) - \mathcal{E}_\theta(\rho) = \left[\sum_i (\partial_\theta K_{i,\theta}) \rho K_{i,\theta}^\dagger + \sum_i K_{i,\theta} \rho (\partial_\theta K_{i,\theta}^\dagger)\right] d\theta + O((d\theta)^2)$$

SUFFICIENCY: QEC PROTOCOL. (CONT'D)

$R \circ \mathcal{E}_\theta(\rho) = \rho$ due to the QEC condition

$\rho = P\rho P = P\rho$

$$R(\mathcal{E}_{\theta+d\theta}(\rho) - \mathcal{E}_\theta(\rho)) = \Big[\sum_i P U_i^\dagger (\partial_\theta K_{i,\theta}) \rho \, P K_i^\dagger U_i P + h.c.\Big] d\theta + O(d\theta^3)$$

$$= \Big[\sum_i P U_i^\dagger (\partial_\theta K_{i,\theta}) \rho \, P U_i \sqrt{\lambda_i} U_i P + h.c.\Big] d\theta + O(d\theta^3)$$

$$= \Big[\sum_i P U_i^\dagger \sqrt{\lambda_i} (\partial_\theta K_{i,\theta}) \rho P + h.c.\Big] d\theta + O(d\theta^3)$$

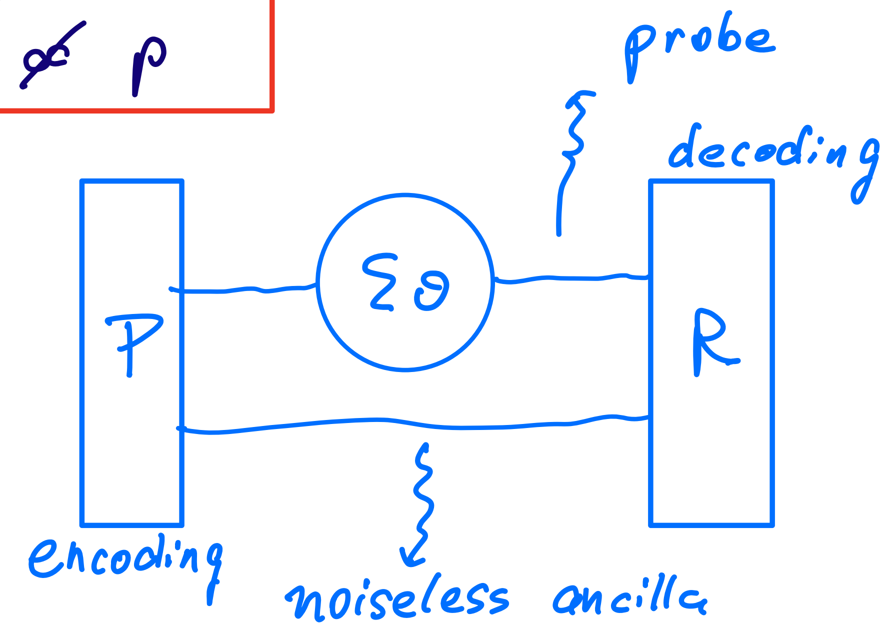
$$= \Big[\sum_i P \underbrace{K_i^\dagger (\partial_\theta K_{i,\theta})}_{-iH_\theta} \rho P + h.c.\Big] d\theta + O(d\theta^3)$$

$$= P \, -i[H_\theta, \rho] P \, d\theta + O(d\theta^3)$$

$$= -i [\underbrace{P H_\theta P}_{H_L \not\propto \mathbb{I}_L}, \rho] \, d\theta + O(d\theta^2)$$

SUFFICIENCY: QEC PROTOCOL. (CONT'D)

* We need to find P s.t. $\boxed{\begin{cases} P K_{i,\theta}^\dagger K_{j,\theta} P \propto P \\ P H_\theta P \not\propto P \end{cases}}$

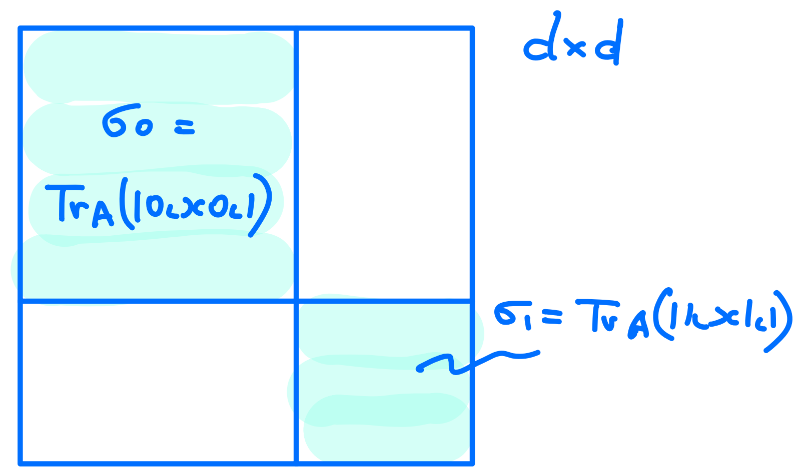


* Noiseless ancilla helps:

$$\begin{cases} \langle 0_L | K_{i,\theta}^\dagger K_{j,\theta} \otimes \mathbb{1}_A | 1_L \rangle = 0 \qquad \forall i,j \\ \langle 0_L | K_{i,\theta}^\dagger K_{j,\theta} \otimes \mathbb{1}_A | 0_L \rangle = \langle 1_L | K_{i,\theta}^\dagger K_{j,\theta} \otimes \mathbb{1}_A | 1_L \rangle \\ \langle 0_L | H_\theta \otimes \mathbb{1}_A | 0_L \rangle \neq \langle 1_L | H_\theta \otimes \mathbb{1}_A | 1_L \rangle \end{cases}$$

$$\Longleftarrow \qquad \mathrm{Tr}\left((\sigma_0 - \sigma_1) K_{i,\theta}^\dagger K_{j,\theta} \right) = 0 \qquad \forall i,j$$

$$\mathrm{Tr}\left((\sigma_0 - \sigma_1) H_\theta \right) \neq 0$$



$$\Longleftarrow \qquad H_\theta \notin \mathrm{span}\{ K_{i,\theta}^\dagger K_{j,\theta}, \ \forall i,j \}$$

THEOREM STATEMENT — REVISITED

* $\exists$ parallel (or sequential strategy), s.t.

$$F^{(par)} \; (\text{or } F^{(seq)}) = \Theta(n^2)$$

IF AND ONLY IF $\quad H_\theta \notin \mathrm{span}\{K_{i,\theta}^\dagger K_{j,\theta}, \; \forall i,j\}$ (HNKS)

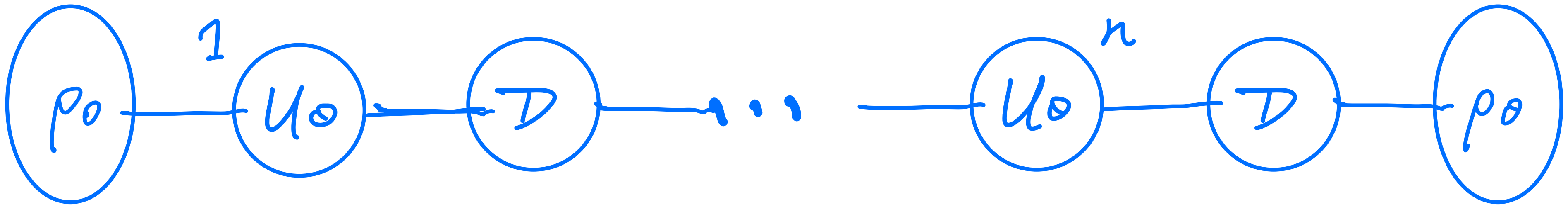
* NECESSITY: $\quad F \leq \|\alpha\| \, O(n) + \|\beta\| \, O(n^2)$

$$\nexists h, \beta = 0 \iff \text{HNKS}$$

* SUFFICIENCY: $\quad$ HNKS + Noiseless ancilla $\Longrightarrow$

$$\exists P, \text{ s.t. } \begin{cases} P K_{i,\theta}^\dagger K_{j,\theta} P \propto P \\ P H_\theta P \not\propto P \end{cases} \Longrightarrow \text{Effective unitary evolution}$$

NOISY PHASE ESTIMATION UNDER BIT-FLIP NOISE



* : Bit-flip noise

* : Input state : $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$

QFI $= O(1)$, if no control

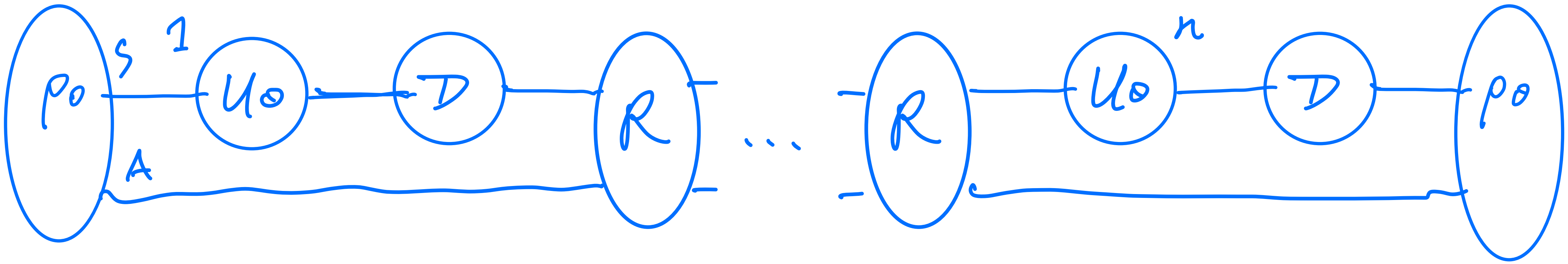
* HNKS: $\mathcal{E}_\theta = D \circ U_\theta$, $\begin{cases} K_1 = \sqrt{1-p}\, e^{-iZ\theta} \\ K_2 = \sqrt{p}\, X e^{-iZ\theta} \end{cases}$

Hamiltonian: $H = i K_1^\dagger \partial_\theta K_1 + i K_2^\dagger \partial_\theta K_2 = K_1^\dagger K_1 \cdot Z + K_2^\dagger K_2 \cdot Z = Z$

Kraus span: span$\{ \mathbb{I}, X \}$ ✓ HNKS satisfied, HL achievable

For dephasing noise, however, $Z \in \text{span}\{\mathbb{1}, Z\}$ × HNKS violated, HL not achievable

NOISY PHASE ESTIMATION UNDER BIT-FLIP NOISE



* QEC is possible, using noiseless ancilla

* Repetition code is sufficient:

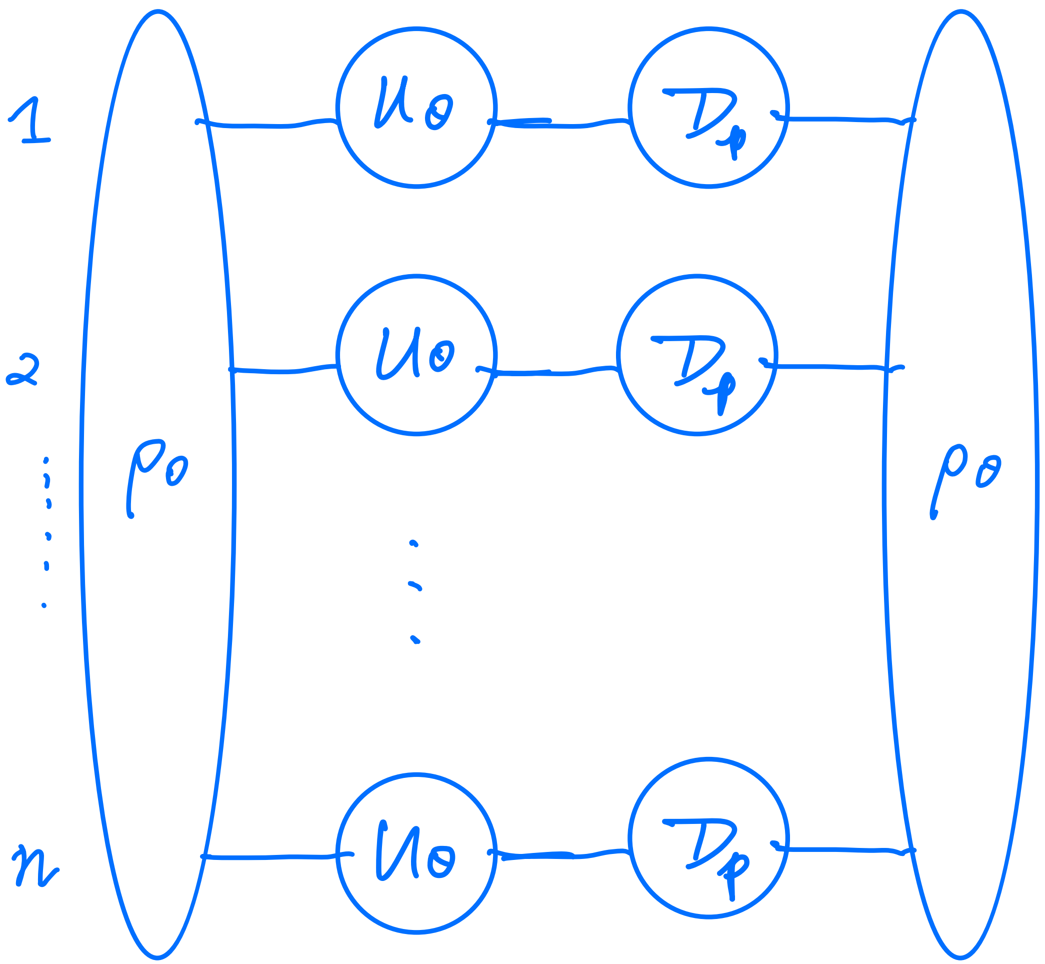
  $|0_L\rangle = |0_S\rangle|0_A\rangle$, $|1_L\rangle = |1_S\rangle|1_A\rangle$

* Detection : $\begin{cases} P_1 = |00\rangle\langle 00| + |11\rangle\langle 11| & \dots \text{ No. error} \\ P_2 = |01\rangle\langle 01| + |10\rangle\langle 10| & \dots \text{ bit-flip error on } S \end{cases}$

* Correction: $\begin{cases} U_1 = \mathbb{1} \otimes \mathbb{1} \\ U_2 = X \otimes \mathbb{1} \end{cases}$

* $R \circ [(\mathcal{D} \circ U_\theta) \otimes \mathbb{1}_A] \sim U_\theta \otimes \mathbb{1}_A$, in the code subspace.

NOISY PHASE ESTIMATION UNDER BIT-FLIP NOISE



* n-qubit GHZ state + Bit-flip noise

QFI $= 4n^2$, as in the noiseless case

* QEC interpretation [with Caveat]

* Repetition Code:

$|0_L\rangle = |0 0 \cdots 0\rangle \quad |1_L\rangle = |1 1 \cdots 1\rangle$

* QEC condition:

Errors: $X^{t_1} \otimes \cdots \otimes X^{t_n}$, $|t| \le \lfloor (n-1)/2 \rfloor$

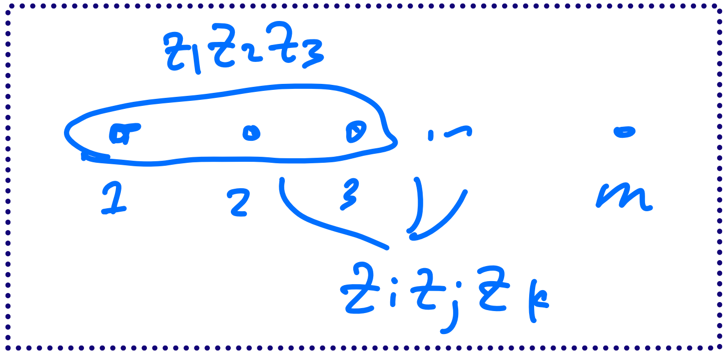
We want $\langle 0 \cdots 0| K_i^\dagger K_j |1 \cdots 1\rangle = 0$

$\langle 0 \cdots 0| X^{s_1} \otimes \cdots X^{s_n} |1 \cdots 1\rangle = 0$ if weight$(s) \le n-1$,

* We can tolerate $\lfloor n-1/2 \rfloor$ number of bit-flip noises

* The probability of getting $> \lfloor n-1/2 \rfloor$ number of noises is exponentially small in n (when $p < 1/2$)

ZZZ INTERACTION UNDER SINGLE-QUBIT NOISE



code block

* $U_\theta = \exp(-iH\theta)$,

  $H = \sum_{ijk} Z_i \otimes Z_j \otimes Z_k$, ijk distinct

  $\#(ijk): \binom{m}{3} = \frac{m(m-1)(m-2)}{6}$

* $$\boxed{\mathcal{D}(\cdot) = \sum_i p_{i,x} X_i (\,) X_i + \sum_i p_{i,y} Y_i (\,) Y_i + \sum_i p_{i,z} Z_i (\,) Z_i}$$

  not i.i.d. noise

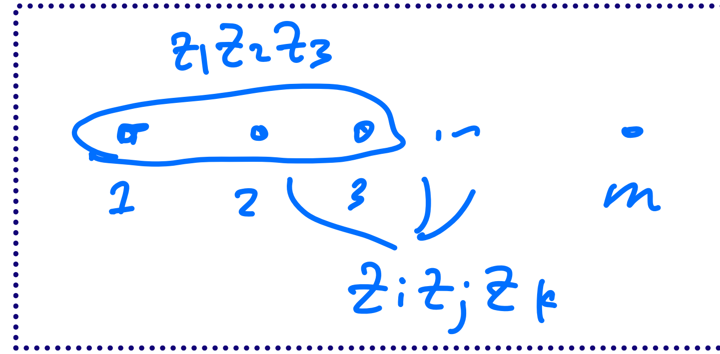
ground state　　highest excited state

* Noiseless case: Use GHZ: $\frac{|0\cdots 0\rangle + |1\cdots 1\rangle}{\sqrt{2}}$

  $Z_i Z_j Z_k |0\cdots 0\rangle = |0\cdots 0\rangle$; $Z_i Z_j Z_k |1\cdots 1\rangle = -|1\cdots 1\rangle$.

  $\mathrm{QFI} = \underline{4 \cdot \binom{m}{3}^2 \cdot n^2}$

ZZZ INTERACTION UNDER SINGLE-QUBIT NOISE



code block

* $U_\theta = \exp(-iH\theta)$,

  $H = \sum_{ijk} Z_i \otimes Z_j \otimes Z_k$, ijk distinct

  $\#(ijk): \binom{m}{3} = \frac{m(m-1)(m-2)}{6}$

* Noisy case: $\{|0\cdots0\rangle, |1\cdots1\rangle\}$ is not a good QEC code,

  e.g. $\langle 0\cdots0|Z_1|0\cdots0\rangle \neq \langle 1\cdots1|Z_1|1\cdots1\rangle$

* Use (generalized) shor code

Codewords: $\left(\frac{|000\rangle \pm |111\rangle}{\sqrt{2}}\right)^{\otimes 3} \longrightarrow \left(\frac{|0\cdots0\rangle \pm |1\cdots1\rangle}{\sqrt{2}}\right)^{\otimes 3} =: \begin{cases} |+_R +_R +_R\rangle \\ |-_R -_R -_R\rangle \end{cases}$

Shor code   (generalized) Shor code:

ZZZ INTERACTION UNDER SINGLE-QUBIT NOISE

* $$\left(\frac{|0\cdots0\rangle \pm |1\cdots1\rangle}{\sqrt{2}}\right)^{\otimes 3} \xrightarrow{X \text{ noise}} \left(\frac{|010\cdots0\rangle \pm |101\cdots1\rangle}{\sqrt{2}}\right)\left(\frac{|0\cdots0\rangle \pm |1\cdots1\rangle}{\sqrt{2}}\right)^{\otimes 2}$$

Can correct single-qubit noise

$$\xrightarrow{Z \text{ noise}} |-\rangle_R |+\rangle_R |+\rangle_R$$

* QFI:

$1 \;\cdots\; m/3$	$m/3+1 \;\cdots\; 2m/3$	$2m/3+1 \;\cdots\; m$	
$(\bullet \;\bullet\; \cdots \;\bullet)$	$(\bullet \;\bullet\; \cdots \;\bullet)$	$(\bullet \;\bullet\; \cdots \;\bullet)$	Logical / Noise
Z_i	Z_j	Z_k	X_L
$Z_i Z_j$	Z_k		= single Z noise
$Z_i Z_j Z_k$			= single Z noise

* QFI: $4n^2 \cdot \left(\frac{m}{3}\right)^6 < 4n^2 \left(\frac{m}{3}\right)^3$ noiseless QFI

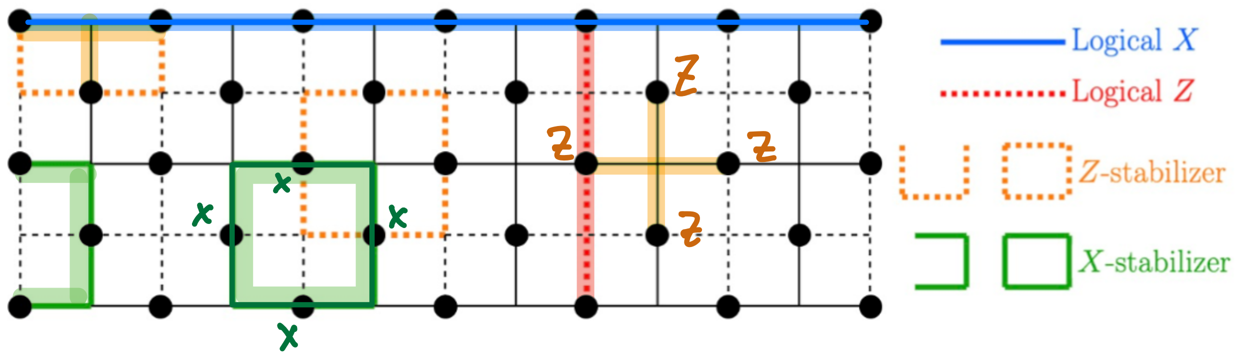
$= O(n^2 m^6)$

* Recover noiseless QFI up to constant factor

ZZZ INTERACTION UNDER SINGLE-QUBIT NOISE

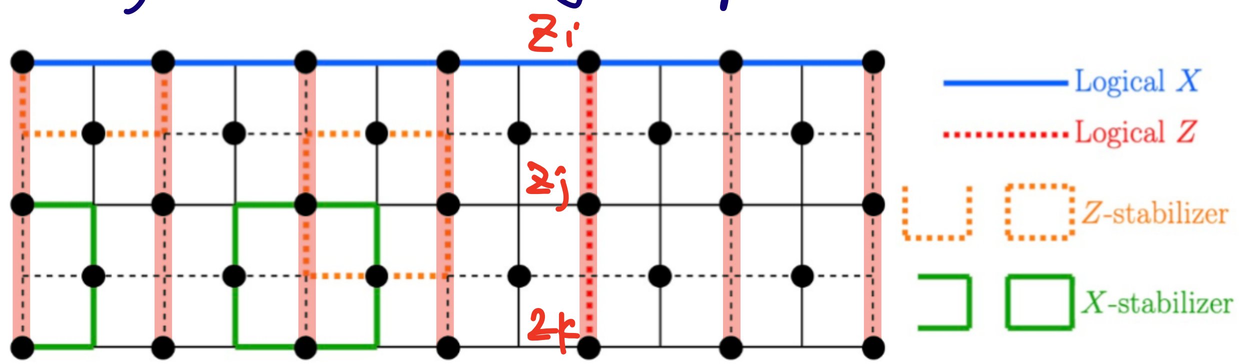
* Use (thin) surface code

Stabilizers:



Any $E_i^\dagger E_j$, where $E_{i,j}$ is single-qubit, anti-commutes with some stabilizer.

* Number of $Z_i Z_j Z_k$ that is logical operator : $\propto m$



* QFI $\sim O(n^2 m^2) < O(n^2 m^6)$, [suboptimal factor]

SUMMARY OF PART 4

* HNKS condition $\Longleftrightarrow$ $F = \Theta(n^2)$

* Necessity: Channel QFI upper bounds

* Sufficiency: QEC protocol using noiseless ancilla

* phase estimation under bitflip noise : repetition code
  - Full recover noiseless behaviour

* ZZZ interaction within m-qubit code blocks
  - Repetition code fails to achieve HL
  - Generalized Shor code recovers optimal m-scaling
  - Thin surface code recovers suboptimal m-scaling

PART 5

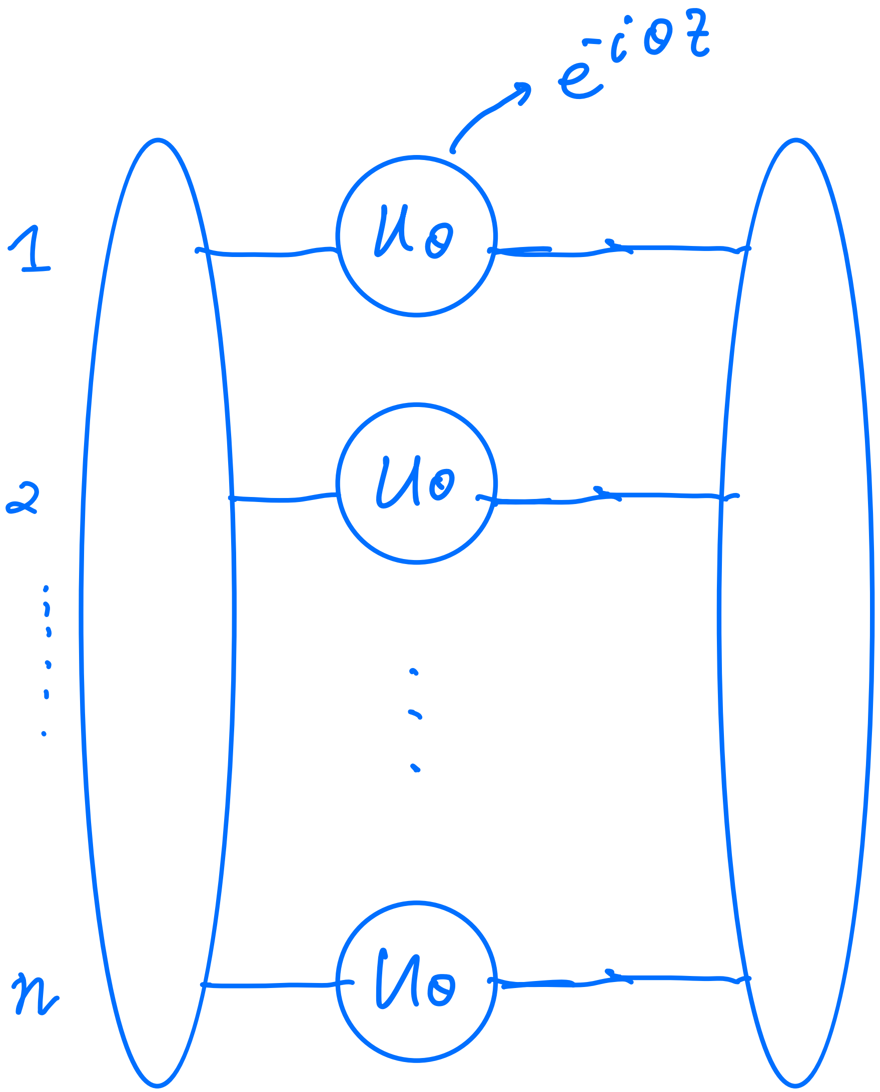
FAULT TOLERANCE IN QUANTUM METROLOGY

— PHASE ESTIMATION UNDER BIT-FLIP NOISE

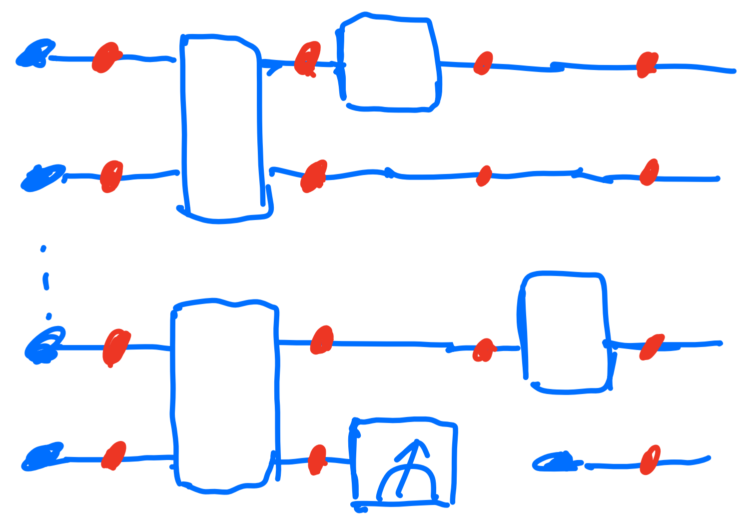
* SETTING
* MEASUREMENT
* STATE PREPARATION

* Plot of Fisher information

SETTING



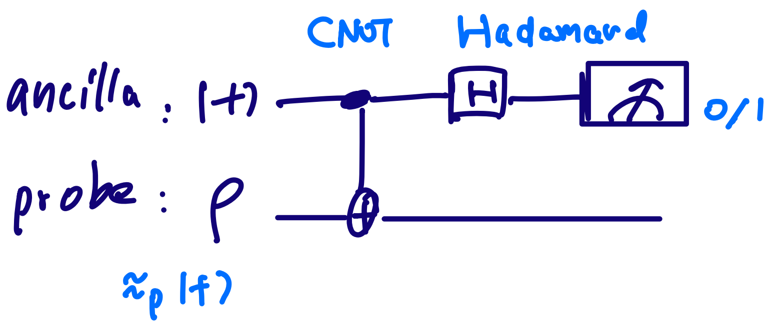
* CIRCUIT-LEVEL NOISE
  - Initialization : $|+\rangle$, $|0\rangle$
  - Measurement : $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$
  - Gate and idle error: Bit-flip noise, X
* NOISE PROBABILITY : p



STATE PREPARATION —— SINGLE QUBIT INITIALIZATION

* Goal : Preparing $|+\rangle^{\otimes n}$

* Single-qubit Initialization error:



$\Longleftrightarrow$ non-constructive projection of ρ to $|\pm\rangle$, with error $\sim p$

CNOT $|0\rangle|\psi\rangle \to |0\rangle|\psi\rangle$, CNOT $|\psi\rangle|+\rangle \to |\psi\rangle|+\rangle$, $H|0\rangle \to (|0\rangle+|1\rangle)/\sqrt{2}$

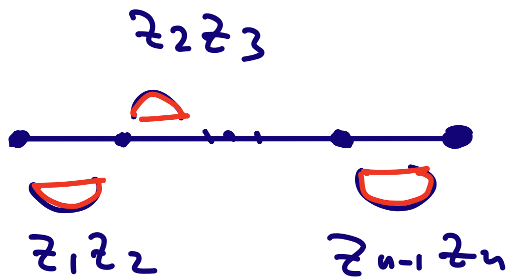
CNOT $|1\rangle|\psi\rangle \to |1\rangle X|\psi\rangle$, CNOT $|\psi\rangle|-\rangle \to Z|\psi\rangle|-\rangle$. $H|1\rangle \to (|0\rangle-|1\rangle)/\sqrt{2}$

Outcome 0 if ρ is $|+\rangle$, 1 if ρ is $|-\rangle$.

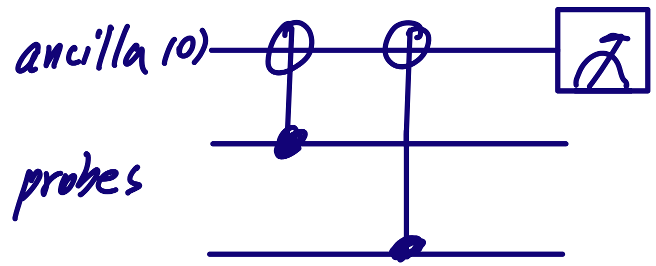
* Repeat measurement $\sim O(\log n)$ times, error prob $\xrightarrow{n} 0$

STATE PREPARATION —— LOGICAL-STATE PREPARATION

* Goal : Preparing $(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2}$ from $|+\rangle^{\otimes n}$

* 

(Stabilizers)

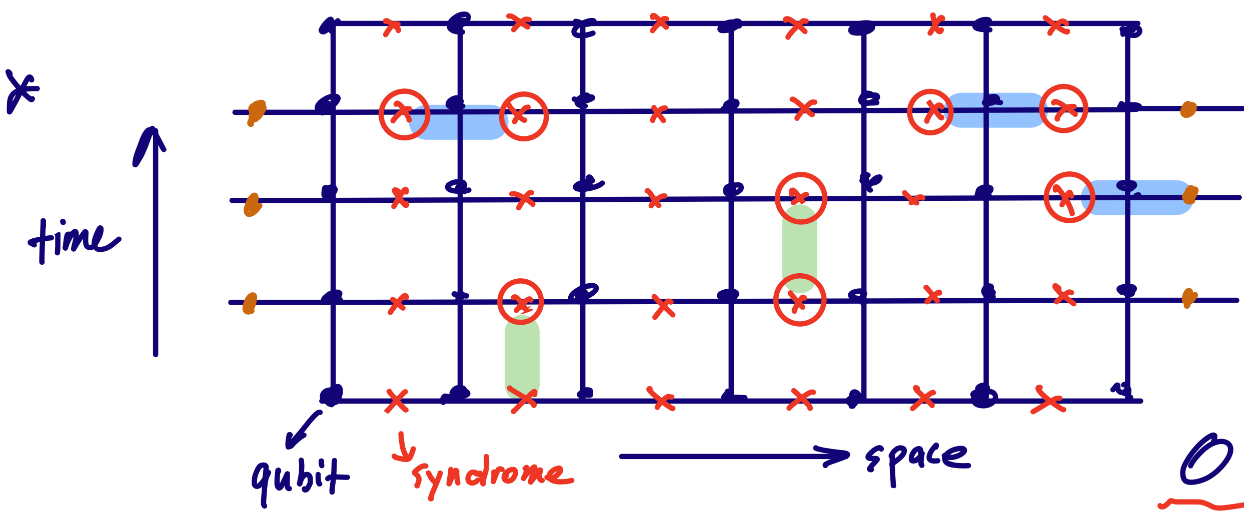


"ZZ measurement" with error $\sim p$

* Start with $|+\rangle^{\otimes n}$, Measure Z_1Z_2, Z_2Z_3, $\cdots$ $Z_{n-1}Z_n$, Perform Correction

$\Longrightarrow$ Output state must be logical state, Stabilized by $X_L = X_1 X_2 \cdots X_n$

$\Longrightarrow$ Output state must be GHZ $= (|0_L\rangle + |1_L\rangle)/\sqrt{2}$

* 

○ detector : syndrome change from previous round

Bit-flip error on qubits

syndrome measurement errors

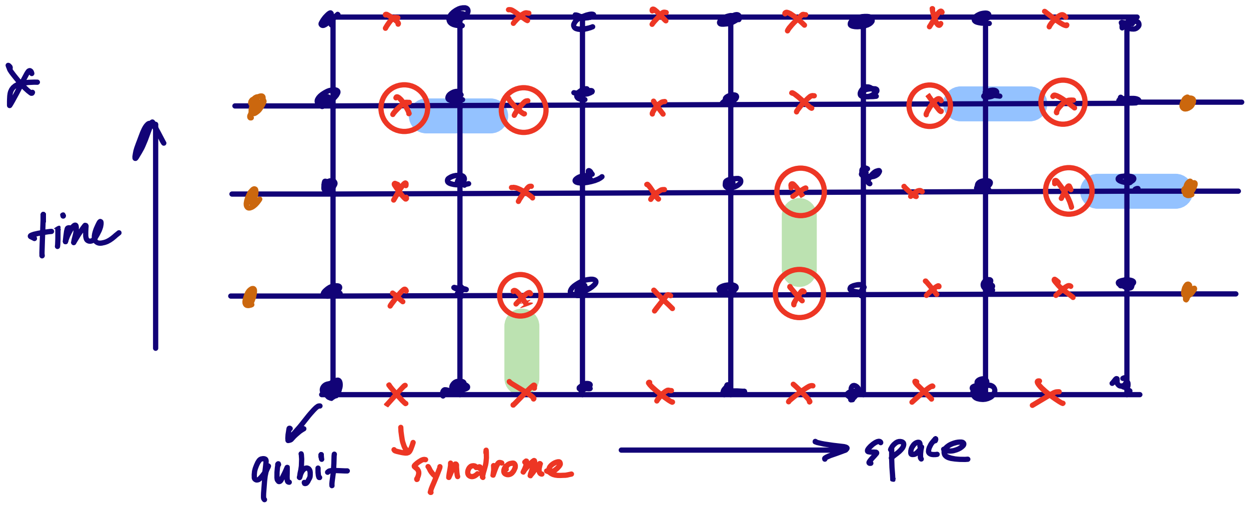
Decoder : Minimum-Weight Perfect Matching (MWPM)

$O(\log n)$ repetitions is sufficient for our purpose

STATE PREPARATION —— LOGICAL-STATE PREPARATION (CONT'D)

* Goal: Preparing $(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2}$ from $|+\rangle^{\otimes n}$

*



○ detector: syndrome change from previous round

Bit-flip error on qubits

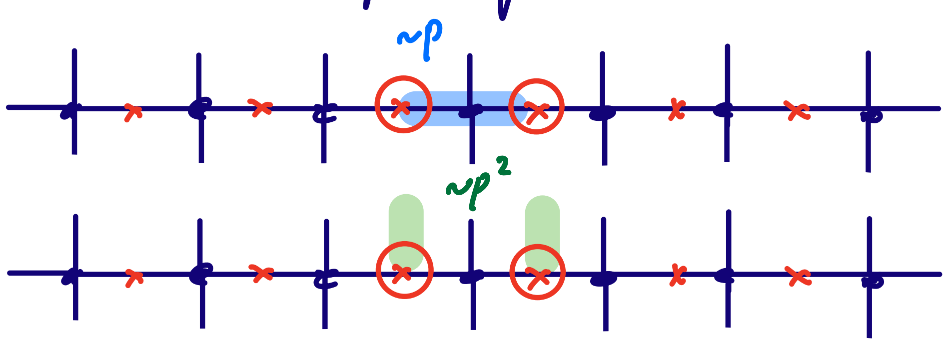
syndrome measurement errors

Decoder: Minimum-Weight Perfect Matching (MWPM)

* $(|\vec{0} \oplus \vec{b}\rangle + |\vec{1} \oplus \vec{b}\rangle)/\sqrt{2}$

Magnetization: $n - 2\,\mathrm{wt}(b)$, b: # of bit-flip errors

* errors happening in the end cannot fully corrected, $\mathrm{wt}(b) \sim n \cdot p^2$



cannot distinguish

STATE PREPARATION ——— S GATE

* $\dfrac{|0\rangle^{\otimes n} + |1\rangle^{\otimes n}}{\sqrt{2}} \longrightarrow \dfrac{|0\rangle^{\otimes n} + i\,|1\rangle^{\otimes n}}{\sqrt{2}}$

* Perform S gate $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ randomly on a single qubit

* Reason: (Assume no error) $\boxed{\text{When } \theta \approx 0,}$

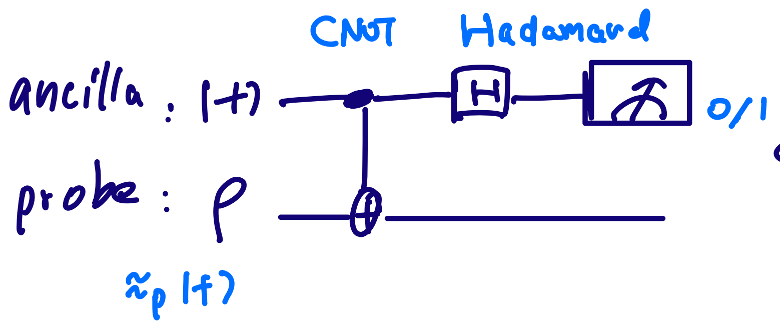
Before S gate : $p(\text{even}) = \cos^2(2n\theta) \approx 1$, $p(\text{odd}) = \sin^2(2n\theta) \approx 0$

After S gate : $p(\text{even}) = \dfrac{1 - \sin(2n\theta)}{2} \approx 1/2$, $p(\text{odd}) = \dfrac{1 + \sin(2n\theta)}{2} \approx 1/2$

We want a balanced probability distribution, which is more robust against measurement noise

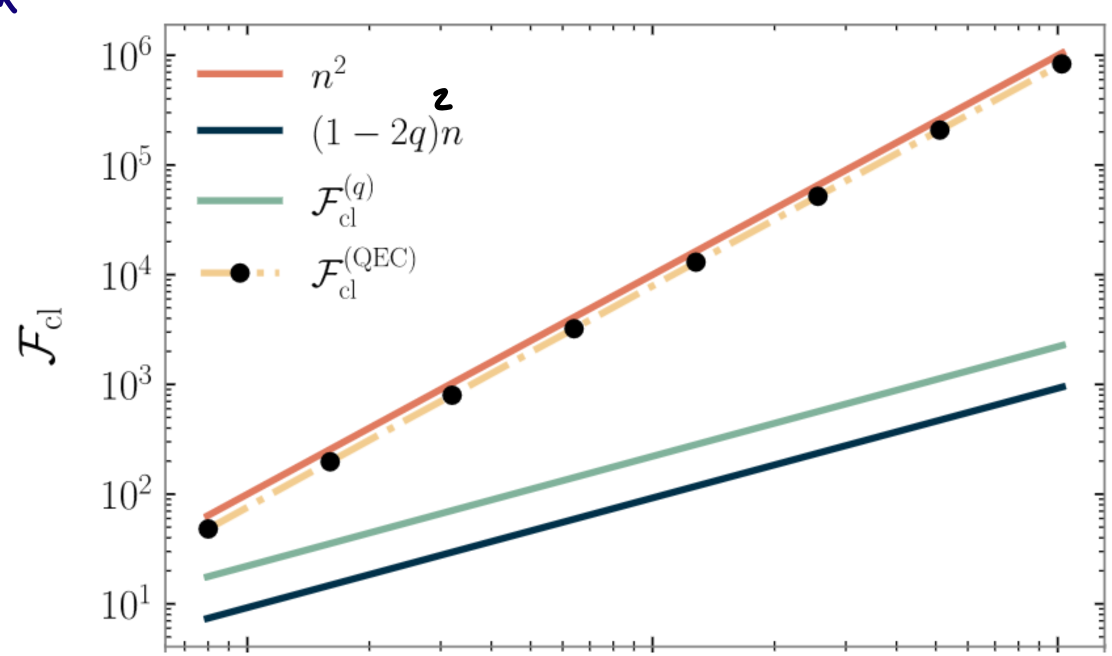
MEASUREMENT NOISE + PLOT OF FI

* Non-destructive X measurement, repeat $O(\log n)$ times on all qubits

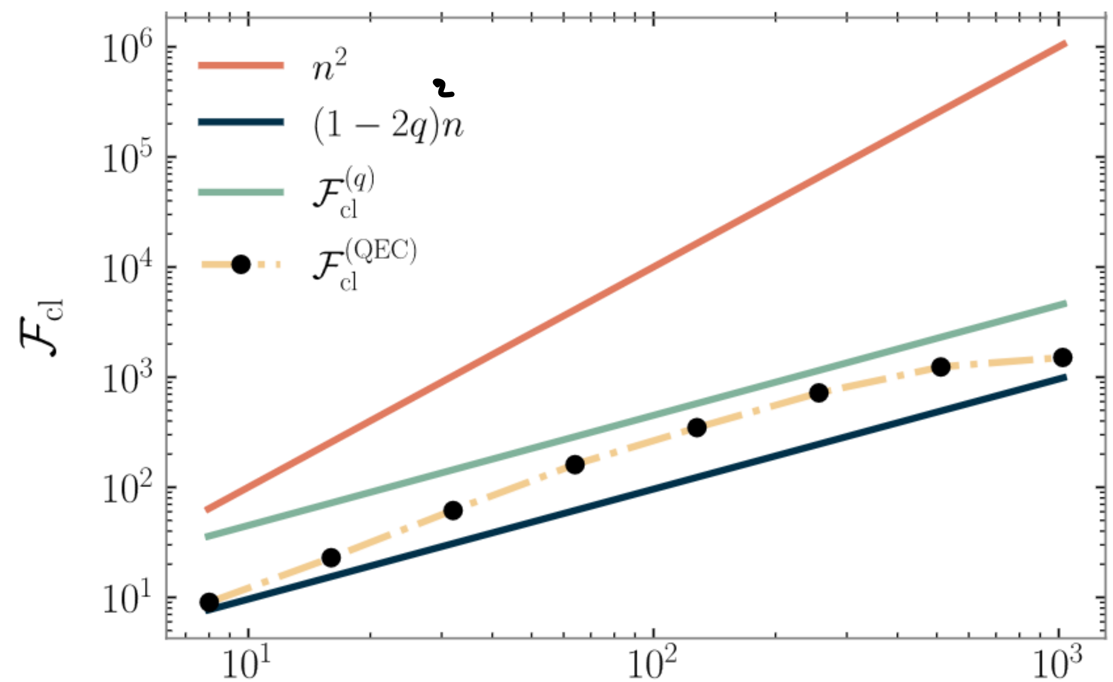


$\Longleftrightarrow$ non-constructive projection of ρ to $|\pm\rangle$, with error $\sim p$

* FI plots :



below threshold



above n threshold.

SUMMARY OF PART 5

* FT protocol for achieving the HL in phase estimation under bit-flip noise

* Robust state preparation: Repeated Stabilizer Measurements $\Rightarrow$ Threshold

* Robust measurement: Repeated Non-destructive Measurements

* S gate manipulation